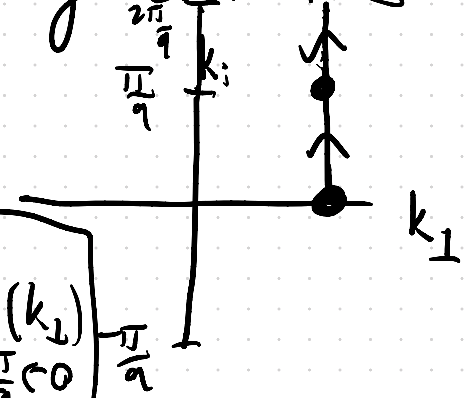
Lecture 22

Last time: Inversion symmetry constraints on Wilson loop



for $k_\perp = -k_\perp + \vec{b}_\perp$ $\quad \vec{b}_\perp \in \check{T}$

$$\boxed{W_{\frac{2\pi}{a}\leftarrow 0}(k_\perp) = \rho_{(0,k_\perp)}(I)\, W^\dagger_{\frac{\pi}{a}\leftarrow 0}(k_\perp)\, \rho_{(\frac{\pi}{a},k_\perp)}(I)\, W_{\frac{\pi}{a}\leftarrow 0}(k_\perp)}$$

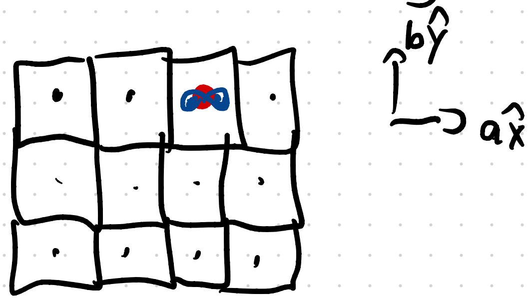
$$\rho^{nm}_k(I) = \vec{u}^\dagger_{nk} \cdot V(Ik-k)\, B(I)\, \vec{u}_{mk}$$

is the group representation matrix for I

$$\det W = e^{i\sum_{l=1}^{n_{occ}} \varphi_l(k_\perp)} = e^{i\pi\left(n_-^{(0,k_\perp)} + n_-^{(\frac{\pi}{a},k_\perp)}\right)}$$

n_-^k - # of negative eigenvalues of $e_k(I)$

Lets extend this to 2D $\qquad \vec{k} \to (k_x, k_y)$



two band NN tight binding model

1D

$$h(k_x) = (\Delta + t_1 \cos k_x a)\sigma_z + t_2 \sin k_x \sigma_y$$

$\downarrow$ 2D

$$h(k_x, k_y) = (\Delta + t_1 \cos k_x a)\sigma_z + t_2 \sin k_x \sigma_y$$

This is boring

Two phases of the 1D chain

1 $\Delta > t_1 > 0 \quad \varphi = 0$

2 $0 < \Delta < t_1 \quad \varphi = \pi$

$\boxed{h(k_x, k_y = 0)}$ — in phase 1

$h(k_x, k_y = \frac{\pi}{b})$ — in phase 2

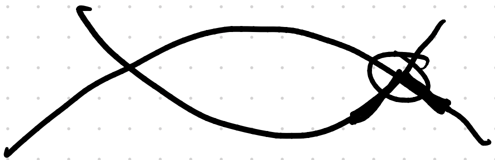
$\Delta > t_1 > 0$

$$h(k_x, k_y) = (\Delta + \Delta \cos k_y b + t_1 \cos k_x a)\sigma_z + t_2 \sin k_x a \sigma_y$$

$$h(k_x, k_y = 0) = (2\Delta + t_1 \cos k_x a)\sigma_z + t_2 \sin k_x a \sigma_y \quad 1$$

$$h(k_x, k_x = \frac{\pi}{b}) = t_1 \cos k_x a \sigma_z + t_2 \sin k_x a \sigma_y \quad 2$$

$$\varepsilon_\pm = \pm\sqrt{(\Delta+\Delta\cos k_y b + t_1\cos k_x a)^2 + t_2^2 \sin^2 k_x a}$$



$k_x = \frac{\pi}{a}$

$\cos k_y b = 1 - t_1/\Delta \rightarrow \varepsilon_+ = \varepsilon_- = 0$

To fix this add an extra term to h

$+ f(\vec{k})\sigma_x$

$$\varepsilon_\pm = \pm\sqrt{(\Delta+\Delta\cos k_y b + t_1\cos k_x a)^2 + t_2^2 \sin^2 k_x a + f(k)^2}$$

$B(I) = \sigma_z$

$B(T) = \sigma_0 \mathcal{K}$

Inversion:

$$\sigma_z f(\vec{k}) \sigma_x \sigma_z = f(-k) \sigma_z$$

$$\Rightarrow \quad f(k) = -f(-k)$$

TRS

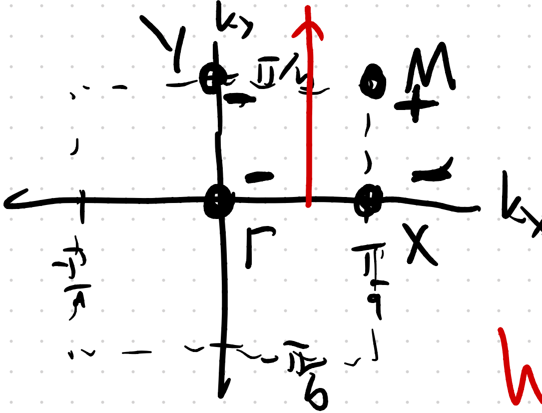
Need to give up TRS for $f \neq 0$

$$\cancel{f^*(k) = f(k) = f(-k)}$$

$$f(\vec{k}) = -t_2 \sin k_y b$$

$$h^{2D}(k_x, k_y) = \left[\Delta(1 + \cos k_y b) + t_1 \cos k_x a\right] \sigma_z + t_2 \sin k_x a \, \sigma_y - t_2 \sin k_y a \, \sigma_x$$

Gapped $\forall k$ when $\Delta > t_1 > 0$, $t_2 \neq 0$



I-eigenvalues of
− energy band

Consider Wilson loops

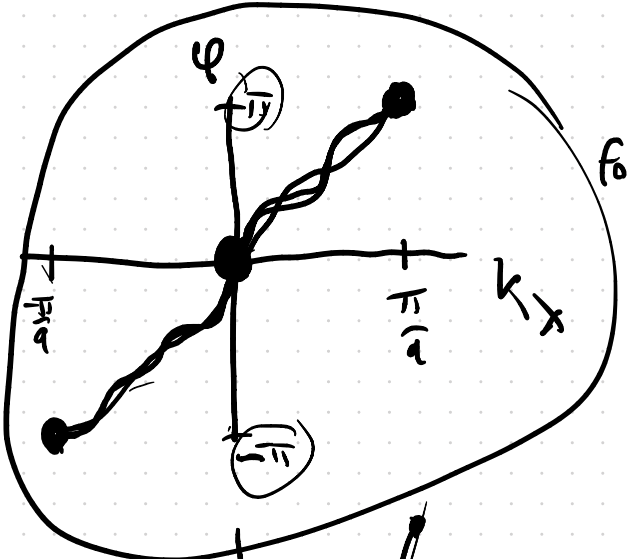
$$W_{\frac{2\pi}{b}\leftarrow 0}(k_x) = u^{\dagger}_{-,(k_x,0)} V\left(\tfrac{2\pi}{b}\hat{y}\right) \prod_{k_y}^{\frac{2\pi}{b}\leftarrow 0} P(k_x,k_y) \cdot u_{-,(k_x,0)}$$

at $k_x = 0$

$$\det W_{\frac{2\pi}{b}\leftarrow 0}(k_x=0) = e^{i\pi 2} = 1 = e^{i\varphi_-(k_x=0)}$$

$$\therefore \varphi_-(k_x=0) = 0 \mod 2\pi$$

at $k_x = \frac{\pi}{a}$



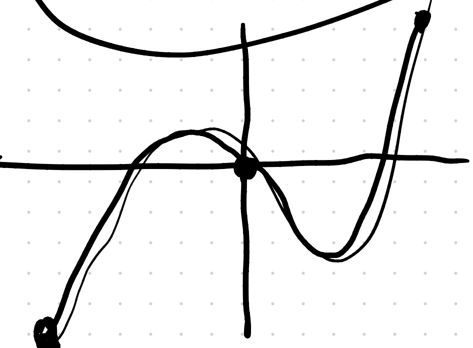
for our model

det $W_{\frac{2\pi}{b}\leftarrow 0}(k_x = \frac{\pi}{a}) = -1$

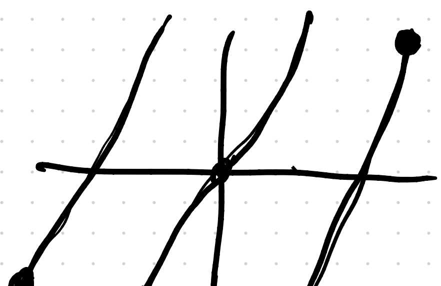
$\varphi_-(k_x = \frac{\pi}{a}) = \pi \mod 2\pi$

$\varphi_-(k_x)$ is related to PyP eigenvals

$PyP|W_{-k_x R_y}\rangle = (R_y + \frac{b}{2\pi}\varphi_-(k_x))|W_{-k_x R_y}\rangle$



Inversion: $\varphi_-(-k_x) = -\varphi_-(k_x)$



define an integer

C - the # of times $\varphi(k_x)$

$$(-1)^C = (-1)^{\sum_{\text{TRIMS}} n_-^k}$$

winds from $-\pi$ to π as k_x goes from $-\frac{\pi}{a} \to \frac{\pi}{a}$

C <u>must</u> be odd for our model

Formula for C

$$e^{i\varphi_-(k_x)} = \det W(k_x) = \det W\left(k_x + \frac{2\pi}{a}\right) = e^{i\varphi_-\left(k_x + \frac{2\pi}{a}\right)}$$

$$\varphi_-\left(k_x + \frac{2\pi}{a}\right) = \varphi_-(k_x) + 2\pi C$$

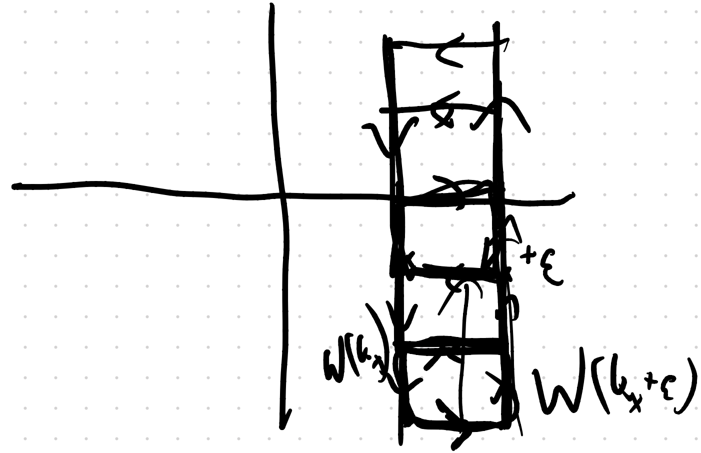
$$\varphi_-(k_x) = C a k_x + f(k_x) \qquad f\left(k_x + \frac{2\pi}{a}\right) = f(k_x)$$

$$\overline{\frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk_x \frac{\partial \varphi}{\partial k_x}} = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk_x \left(Ca + \cancel{\frac{\partial f}{\partial k_x}}\right)^{0}$$

$$= C = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk_x \frac{\partial}{\partial k_x} \operatorname{Im} \log \det W(k_x)$$

$$\frac{\partial}{\partial k_x} \operatorname{Im} \log \det W = \operatorname{Im} \lim_{\varepsilon \to 0} \left[\frac{\log \det W(k_x+\varepsilon) - \log \det W(k_x)}{\varepsilon}\right]$$

$$= \operatorname{Im} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log \det W(k_x+\varepsilon) W^{\dagger}(k_x)$$



Stokes thm

$$\frac{\partial}{\partial k_x} \mathrm{Im} \log \det W = \mathrm{Im} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{\text{squares}} \log \det W_{\square}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{\text{squares}} \oint_{\text{square}} dk \, \mathrm{tr}\, A(\vec{k})$$

$$= \int dk_y \, \mathrm{tr}\, \Omega_{xy}(k_x, k_y)$$

where $\Omega_{xy}(k_x, k_y) \doteq \frac{\partial A_y}{\partial k_x} - \frac{\partial A_x}{\partial k_y} - i\left[A_x, A_y\right]$

$$C = \frac{1}{2\pi} \int dk_x \int dk_y \, \mathrm{tr}\, \Omega_{xy}(k_x, k_y)$$

$$= \frac{1}{2\pi} \oiint d^2k \, \mathrm{tr}\, \Omega_{xy}(k_x, k_y)$$

Chern number $\in \mathbb{Z}$

Consider HWFs $|W_{-}, k_x, R_y\rangle$

$$P_y P |W_{-,k_x,R_y}\rangle = \left[R_y + \frac{b}{2\pi}\varphi(k_x)\right] |W_{-\,k_x,R_y}\rangle$$

$$P_y P |W_{-\,k_x+\frac{2\pi}{a},R_y}\rangle = \left[R_y + \frac{b}{2\pi}\varphi\left(k_x+\frac{2\pi}{a}\right)\right] |W_{-\,k_x+\frac{2\pi}{a},R_y}\rangle$$

$$= \left[(R_y + Cb) + \frac{b}{2\pi}\varphi(k_x)\right] |W_{-\,k_x+\frac{2\pi}{a}\,R_y}\rangle$$

$$\Rightarrow \quad |W_{-\,k_x+\frac{2\pi}{a},R_y}\rangle = |W_{-\,k_x,R_y+Cb}\rangle$$

Consequences

① Apply electric field $\vec{E} = E_0\hat{x}$

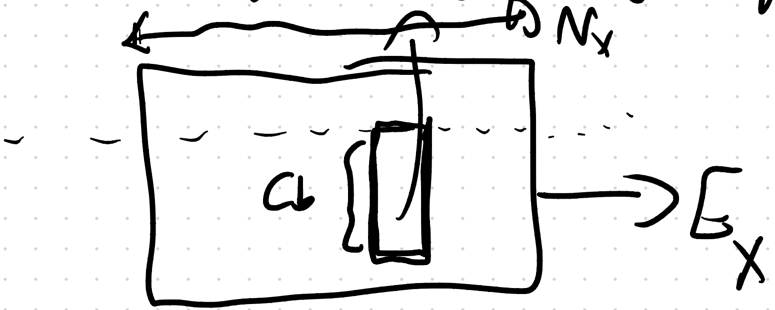
$$\vec{A} = -E_0 t \hat{x} \qquad \vec{E} = -\frac{\partial A}{\partial t}$$

$$\vec{p} \to \vec{p} - q\vec{A} = \vec{p} + qE_0 t \hat{x}$$

$$k \to k(t) = (k_x + qE_0 t, k_y)$$

$$|W_{-k_x, R_y}\rangle \to |W_{-k_x + qE_0 t, R_y}\rangle$$

after one period $\to N_x$ $\quad T = \frac{2\pi\hbar}{aqE_0}$ $\quad |W_{-k_x(t), R}\rangle$ has moved C unit cells in the y direction



$$\Delta Q = C q N_x$$

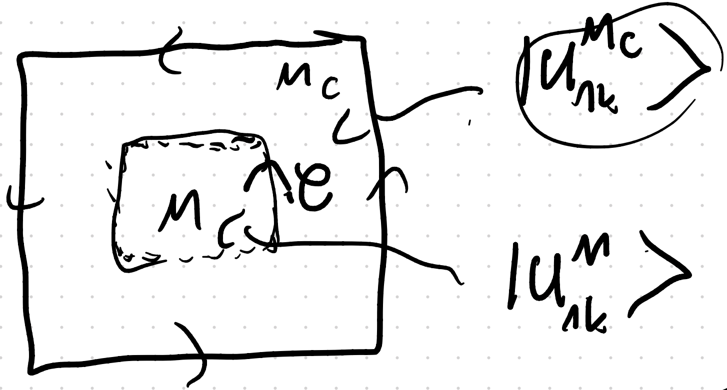
$$\Rightarrow I_y = \frac{\Delta Q}{T} = C q \left(\frac{q}{2\pi\hbar}\right) \overbrace{N_x a E_0}^{V}$$

$$= \underbrace{C \frac{q^2}{h}}_{} V_x$$

G_H - Hall conductance

$\rightarrow$ Integer $\times \frac{q^2}{h}$ for an insulator

Claim $\quad C \neq 0$ the occupied bands do not have exponentially localized Wannier fns



$$C = \frac{1}{2\pi} \oiint d^2k \, \mathrm{tr}\,\Omega = \frac{1}{2\pi}\left[\iint_M d^2k\, \mathrm{tr}\,\Omega + \iint_{M^c} d^2k\, \mathrm{tr}\,\Omega\right.$$

$$= \frac{1}{2\pi}\left[\oint_{\mathcal{C}} \mathrm{tr} A_M \cdot dk - \oint_{\mathcal{C}} \mathrm{tr} A_{M^c}\, dk\right.$$

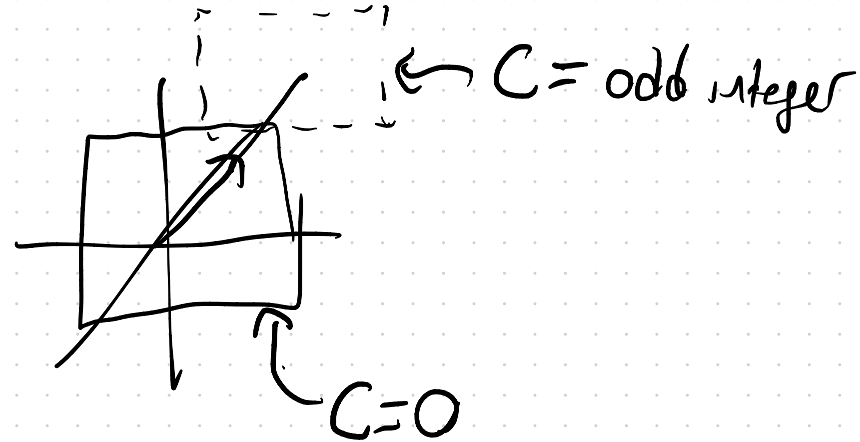
on $\mathcal{C}$ $\quad \mathrm{tr} A_{M^c} = \mathrm{tr} A_M + \nabla\varphi$

$$C = \frac{1}{2\pi} \oint_{\mathcal{C}} \nabla \varphi \cdot d\vec{k}$$

$\Rightarrow$ $C \neq 0$ means I can't choose a single smooth gauge for my Bloch functions

$\rightarrow$ No exponentially localized WFs.

$$|W_{n\vec{R}}\rangle = \frac{v}{(2\pi)^3} \int d^3k \, |\tilde{\psi}_k\rangle e^{-ik\cdot R}$$


C = odd integer
C=0