Lecture 2

Announcements: Office hrs Wednesdays 4-5pm (starting 9/11)

Recap: introduced groups G - set w/ associative "multiplication", identity element, inverses

Ex: Bravais lattice $T = \{ \sum_i n_i \vec{t}_i \mid \vec{t}_i$ linearly independent $n_i \in \mathbb{Z} \}$

- closed under "+"
- $\vec{0} \in T$ is the identity
- $(\sum_i n_i \vec{t}_i)^{-1} = \sum_i (-n_i) \vec{t}_i$

Subgroups: $H \leq G$ is a subgroup if

$H \subset G$ and H is a group

(Note: Every group G has at least two subgroups

- $G \leq G$ - the whole group
- $\{E\} \subset G$ "trivial subgroup")

We saw that the right cosets $Hg_i = \{hg_i \mid h \in H\}$

partition the group

$$G = \bigcup_{i=0}^{n-1} Hg_i \qquad n = |G:H| \quad \underline{\text{index}} \text{ of } H \text{ in } G$$

Define conjugation by $g_i \in G$ $\quad C_{g_i}$

$$C_{g_i}(g) = g_i g g_i^{-1}$$

We say that two elements $g_1, g_2 \in G$ are conjugate if there exists some g_i s.t.

$$C_{g_i}(g_1) = g_i g_1 g_i^{-1} = g_2$$

$$\Rightarrow C_{g_i^{-1}}(g_2) = g_i^{-1} g_2 g_i = g_1$$

$\mathcal{C}(g)$ - conjugacy class of g = {all elements of G conjugate to g}

Conjugacy classes also partition the group

$$a = g_1 b g_1^{-1} \qquad b = g_2 c g_2^{-1}$$

$$a = (g_1 g_2) c (g_1 g_2)^{-1}$$

(Note: not all conjugacy classes are the same size)

Given a subgroup $H \leq G$, we can conjugate H by $g \in G$ $\quad H \to gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

gHg^{-1} is also a subgroup - conjugate subgroup

Def: H is a <u>normal subgroup</u> of G if

$H \leq G$, and $gHg^{-1} = H$ for all $g \in G$

$$\Downarrow$$

$$Hg = gH$$

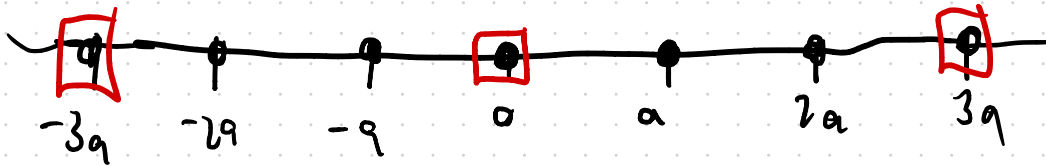
H is a proper normal subgroup of G: $H \triangleleft G$

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

$$H \triangleleft G \Rightarrow ghg^{-1} = h' \in H$$

Example: $T = \{ na\hat{x} \mid n \in \mathbb{Z} \}$ a - lattice constant

$H = \{ 3na\hat{x} \mid n \in \mathbb{Z} \}$



$H \triangleleft T$

$3ma\hat{x} \in H$

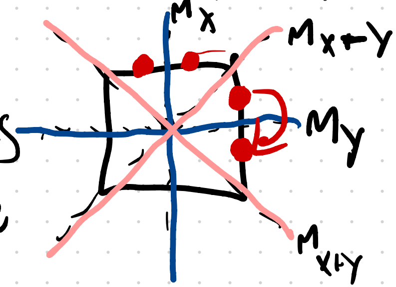
$n\hat{x} \in T$

$$(n\hat{x})^{-1} + (3ma\hat{x}) + (n\hat{x}) = 3ma\hat{x} + n\hat{x} - n\hat{x}$$

$$= 3ma\hat{x}$$

Slightly less trivial example:

Symmetries of a square



C_{4z} – 90° rotation about the origin

$C_{4z}^2 = C_{2z}$: 180° rotation

C_{4z}^3 – 270° rotation

pt group 4mm

$$\{C_{2z}, M_x, M_y, E\} \triangleleft 4mm$$

$$C_{4z} C_{2z} C_{4z}^{-1} = C_{2z}$$

$$C_{4z} M_x C_{4z}^{-1} = M_y$$

$$\{C_{2z}, M_{x+y}, M_{x-y}, E\} \triangleleft 4mm$$

$H \triangleleft G \qquad G = H \cup Hg_1 \cup Hg_2 \cdots \cup Hg_{n-1}$

$\hookrightarrow \{H, Hg_1, Hg_2, \ldots Hg_{n-1}\}$ forms a group!

To see this: $(Hg_i)(Hg_j) = \{h_1 g_i h_2 g_j \mid h_1, h_2 \in H\}$

$$H \triangleleft G \Rightarrow g_i h_2 g_i^{-1} = h_2' \in H$$

$$h_2 = g_i^{-1} h_2' g_i$$

$$(Hg_i)(Hg_j) = \{ h_1 h_2' g_i g_j \mid h_1, h_2' \in H \}$$

$$= Hg_i g_j$$

For normal subgroups, the product of two right cosets is also a right coset

$$(H)(Hg_i) = Hg_i$$

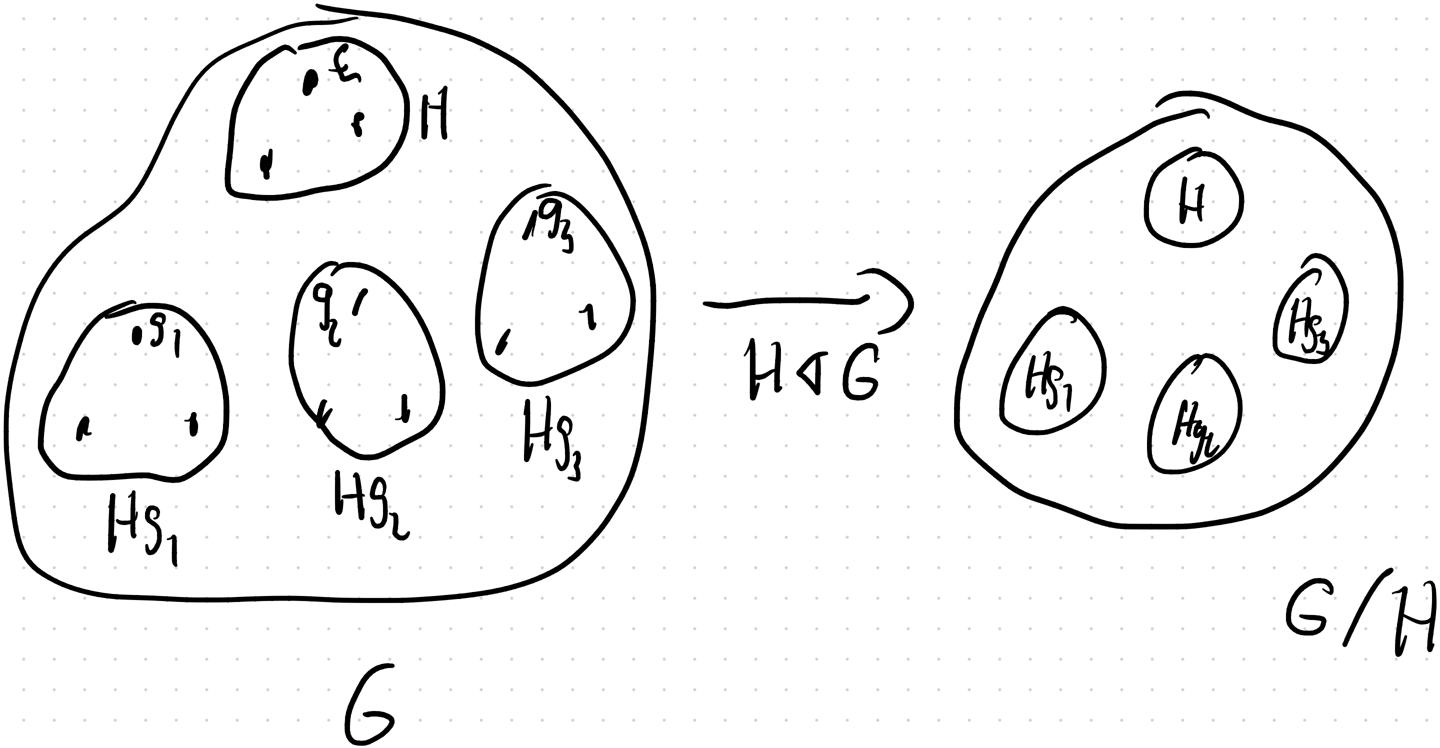
$(Hg_i)(H) = (H)(Hg_i) = Hg_i \leftarrow H$ is an identity element in the set of cosets

for each g_i $\quad g_i^{-1} \in Hg_j$ for exactly one g_j

$$Hg_i^{-1} = Hg_j$$

$$(Hg_i)(Hg_j) = Hg_i Hg_i^{-1} = H$$

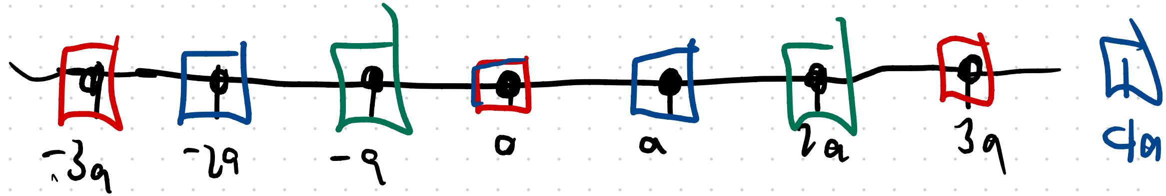
$\rightarrow$ for a normal subgroup $\{Hg_i \mid i = 0, \ldots, n-1\}$ forms a group <u>quotient group</u> G/H



e
H
g_1
g_2
g_3
Hg_1
Hg_2
Hg_3
G
$H \triangleleft G$
H
Hg_1
Hg_2
Hg_3
G/H

Example: $T = \{ n a \hat{x} \mid n \in \mathbb{Z} \}$ a - lattice constant

$H = \{ 3 n a \hat{x} \mid n \in \mathbb{Z} \}$



$H \triangleleft T$

cosets: $H = \{ 0 a \hat{x}, \pm 3 a \hat{x}, \pm 6 a \hat{x} \}$

$H + a\hat{x} = \{ a\hat{x}, 4a\hat{x}, 7a\hat{x}, -2a\hat{x}, -5a\hat{x}, \ldots \}$

$H + 2a\hat{x} = \{ 2a\hat{x}, 5a\hat{x}, \ldots, -a\hat{x}, -4a\hat{x}, \ldots \}$

$$H + (H + a\hat{x}) = H + a\hat{x}$$
$$H + (H + 2a\hat{x}) = H + 2a\hat{x}$$
$$(H + a\hat{x}) + (H + a\hat{x}) = H + 2a\hat{x}$$
$$\underbrace{(H + a\hat{x}) + (H + 2a\hat{x}) = H + 3a\hat{x} = H}_{\text{addition modulo 3}}$$

$H \sim [0]$

$H + a\hat{x} \sim [1]$

$H + 2a\hat{x} \sim [2]$

$T/H \cong \mathbb{Z}_3$ the group of integers w/ addition mod 3

In quantum mechanics:

$$\phi : G \longrightarrow K$$

group of symmetries

unitary operators on space of states

$$\phi : g \to \phi(g) \in K$$

Special subset of functions compatible w/ group multiplication

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \leftarrow \text{group homomorphism}$$

$E_G \in G$ identity in G

$E_K \in K$ identity in K

homomorphism: $\phi(E_G) = E_K$

$\phi(g^{-1}) = [\phi(g)]^{-1}$

Examples: T bravais lattice $\{\sum_i n_i \vec{t}_i \mid n_i \in \mathbb{Z}\}$

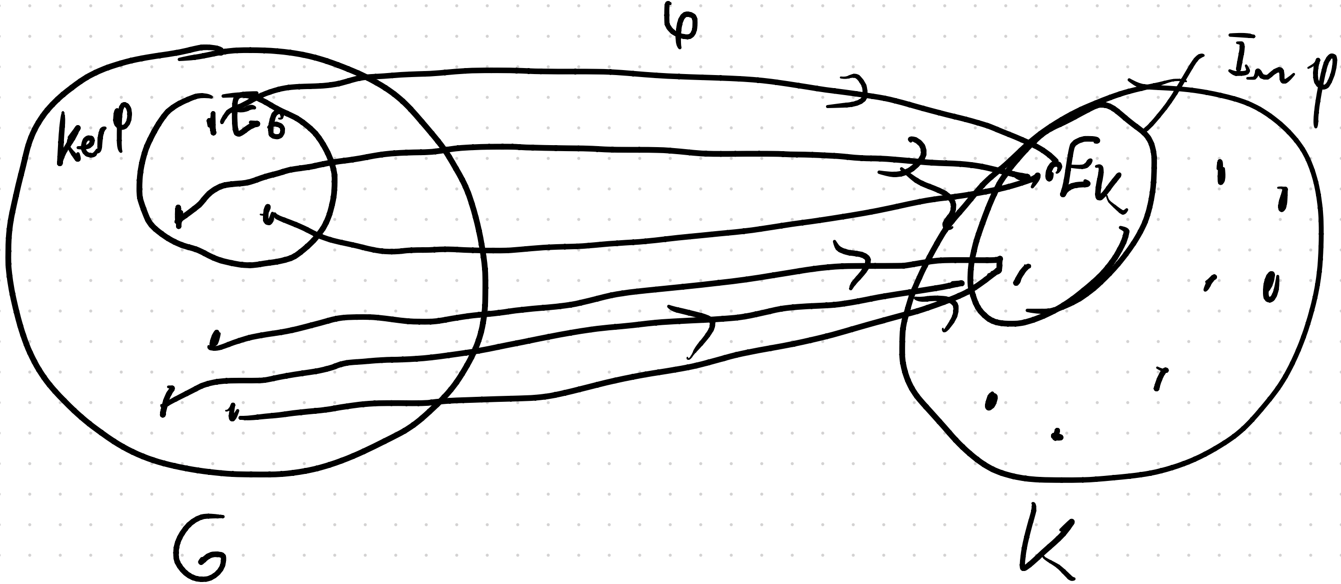
K group of unitary operators on wavefunctions $\psi(\vec{x})$

$$\phi\left(\sum_i n_i \vec{t}\right) = e^{-\frac{i}{\hbar}\sum_i n_i (\vec{p}\cdot\vec{t}_i)}$$ ← unitary operation on wavefunctions

Given $\varphi: G \to K$ a homomorphism,

$$\mathrm{Im}(\varphi) = \{\varphi(g) \mid g \in G\} \subset K$$ image of φ

$$\mathrm{Ker}(\varphi) = \{g \mid \varphi(g) = E_K\} \subset G$$ kernel of φ



① $\mathrm{Im}\,\varphi \leq K$ pf: • $k_1 \in \mathrm{Im}\,\varphi$, $k_2 \in \mathrm{Im}\,\varphi$

$k_1 = \varphi(g_1)$ $k_2 = \varphi(g_2)$

$k_1 k_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) \in \mathrm{Im}\,\varphi$

• $E_K = \varphi(E_G) \in \mathrm{Im}\,\varphi$

• $k = \varphi(g)$, $k^{-1} = [\varphi(g)]^{-1} = \varphi(g^{-1}) \in \operatorname{Im}\varphi$

(2) $\operatorname{Ker}\varphi \trianglelefteq G$ is a normal subgroup

pf (A) $\ker\varphi \trianglelefteq G$

• $g_1, g_2 \in \ker\varphi$

$\varphi(g_1) = \varphi(g_2) = E_K$

$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = E_K$

$\Rightarrow g_1 g_2 \in \ker\varphi$

• $\varphi(E_G) = E_K \Rightarrow E_G \in \ker\varphi$

• $\varphi(g) = E_K \Rightarrow [\varphi(g)]^{-1} = E_K$

$= \varphi(g^{-1}) = E_K$

$\Rightarrow g^{-1} \in \ker \varphi$

To show its normal: $g \in \ker \varphi$

$g' \in G$

$$\varphi(g' g g'^{-1}) = \varphi(g')\varphi(g)\varphi(g')^{-1} = \varphi(g')\varphi(g')^{-1}$$

$= E_K$

$g' g g'^{-1} \in \ker \varphi \Rightarrow \ker \varphi \trianglelefteq G$

First isomorphism theorem: G, K are groups,

$\varphi: G \to K$ is a homomorphism

then $\operatorname{Im}\varphi \cong G/\ker\varphi$

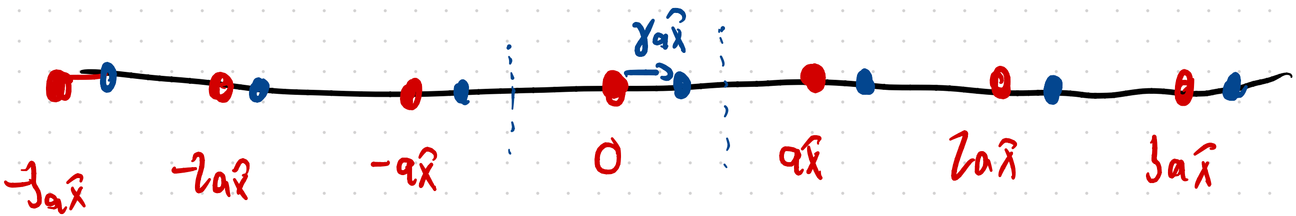
there is a 1-to-1 homomorphism

between right cosets of $\ker\varphi$ and

$\operatorname{Im}\varphi$

Example: $\mathbb{R}^1 = \{\beta a \hat{x} \mid \beta \in \mathbb{R}\}$ $\qquad T \triangleleft \mathbb{R}^1$

$$\underline{T = \{n a \hat{x}} \mid n \in \mathbb{Z}\}$$



$$\mathbb{R}^1 = \bigcup_{\gamma \in (-\frac{1}{2}, \frac{1}{2}]} (T + \gamma a \hat{x})$$

Claim: $\mathbb{R}^1 / T \cong U(1)$ which is the unit circle pf $\varphi(\beta a\hat{x}) = e^{2\pi i \beta} \in U(1)$

$$\varphi(\beta_1 a\hat{x} + \beta_2 a\hat{x}) = e^{2\pi i(\beta_1+\beta_2)} = e^{2\pi i\beta_1} e^{2\pi i \beta_2} = \varphi(\beta_1 a\hat{x})\varphi(\beta_2 a\hat{x})$$

$$\operatorname{Ker}\varphi = \{\eta a\hat{x} \mid e^{2\pi i \eta} = 1\} = T$$

$$\operatorname{Im}\varphi = U(1)$$

$$\mathbb{R}^1/T = U(1)$$ - unit cell of the Bravais lattice T

$H \trianglelefteq G$ $\boxed{E_{G/H}}$

$$G/H = \{\boxed{H}, Hs_1, \dots, Hs_{n-1}\}$$

$\varphi(g) =$ the right coset containing g