Lecture 4

- HW 1 posted - due on gradescope 9/18

  info to sign up on website

- Office Hrs start next week

  Mei: Tuesdays 2-3pm on Zoom

  Derek: Wednesdays 3-4pm 3rd floor ESB

Recap: Representation ρ of a group G

- vector space V
- a group homomorphism $\rho: G \to U(V)$ to unitary operators on V

When are representations the same

Two representations $\rho: G \to U(V)$

$\sigma: G \to U(V)$

ρ and σ are <u>equivalent representations</u> if there is a unitary matrix A such that $\rho \cong \sigma$

$$A\rho(g)A^{\dagger} = \sigma(g) \quad \text{for all } g \in G$$

Now, let G be a group $\rho: G \to U(V)$ a representation

Consider $|v\rangle \in V$ $\quad g \in G$

$\rho(g)|v\rangle$ is also a vector in V

look for subspaces $W \subseteq V$ such that

$\rho(g)|w\rangle \in W$ for all $g \in G$, $|w\rangle \in W$

such a subspace is called an <u>invariant subspace</u>

Given an invariant subspace W, consider its orthogonal complement $W^{\perp} = \{ |w_{\perp}\rangle \in V \mid \langle w_{\perp}|w\rangle = 0 \text{ for all } |w\rangle \in W \}$

$V = W \oplus W^{\perp}$ every $|v\rangle \in V$ can be written as

$|v\rangle = |w\rangle + |w_{\perp}\rangle$

in W in $W^\perp$

if W is an invariant subspace, so is $W^\perp$

pf: Consider $|w\rangle \in W$ and $|w_\perp\rangle \in W^\perp$

for all $g \in G$ $\rho(g)|w\rangle \in W$

$$\Rightarrow \left(\langle w_\perp|\rho(g)|w\rangle\right)^* = 0^*$$

$$\langle w|\rho^\dagger(g)|w_\perp\rangle = 0$$

$$\langle w|\left(\rho(g^{-1})|w_\perp\rangle\right) = 0$$

$$\Rightarrow \rho(g^{-1})|w_\perp\rangle \in W^\perp$$ true for all $g \in G$

$\Rightarrow$ $W^\perp$ is an invariant subspace

Practically speaking: choose a basis for V $\{|v_1\rangle, |v_2\rangle, \ldots\}$

$$\rho(g) = \sum_{ij} |v_i\rangle \underbrace{\langle v_i|\rho(g)|v_j\rangle}_{\rho_{ij}(g)} \langle v_j|$$

matrix acting on column vectors in this basis

$$\rho_{ij}(g) = \begin{pmatrix} \langle v_1|\rho(g)|v_1\rangle & \langle v_1|\rho(g)|v_2\rangle & \cdots \\ \langle v_2|\rho(g)|v_1\rangle & \langle v_2|\rho(g)|v_2\rangle & \cdots \\ \vdots & & \end{pmatrix}$$

using $V = W + W^{\perp}$, we can pick a new basis

$$\{ \underbrace{|w_1\rangle, |w_2\rangle, \ldots}_{\text{Basis for } W}, \underbrace{|w_1^{\perp}\rangle, |w_2^{\perp}\rangle, \ldots}_{\text{Basis for } W^{\perp}} \}$$

$$[\rho(g)] = \begin{array}{cc} W & W^{\perp} \\ \left(\begin{array}{c|c} \langle w_i|\rho(g)|w_j\rangle & \langle w_i|\rho(g)|w_j^{\perp}\rangle \\ \hline \langle w_i^{\perp}|\rho(g)|w_j\rangle & \langle w_i^{\perp}|\rho(g)|w_j^{\perp}\rangle \end{array}\right) \end{array} \begin{array}{c} W \\ W^{\perp} \end{array} = \left(\begin{array}{c|c} \langle w_i|\rho(g)|w_j\rangle & 0 \\ \hline 0 & \langle w_i^{\perp}|\rho(g)|w_j^{\perp}\rangle \end{array}\right)$$

$$\rho(g) \cong \rho_W(g) \oplus \rho_{W^{\perp}}(g) \qquad \rho \cong \rho_W \oplus \rho_{W^{\perp}}$$

$\rho_W : G \to U(W)$
$\rho_{W^\perp} : G \to U(W^\perp)$ are <u>subrepresentations</u> of ρ

If $\rho : G \to U(V)$ has a nontrivial invariant subspace $W \neq V$ or $\{\vec{0}\}$, then ρ is a <u>reducible representation</u>

If ρ is not reducible, it is <u>irreducible</u>

Example: $SU(2)$ spin-$\frac{1}{2}$ representation

$$V_{\frac{1}{2}} = \{ |\uparrow\rangle, |\downarrow\rangle \}$$

$$\rho_{\frac{1}{2}}([\hat{n},\theta]) \to e^{-i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}}$$

this representation is irreducible

consider now two spin $\frac{1}{2}$ particles

$$V = V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} = \{ |\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle \}$$

$$\rho_{\frac{1}{2}\otimes\frac{1}{2}}[(\hat{n},\theta)] = e^{-i\frac{\theta}{2}\vec{\sigma}_1\cdot\hat{n}} \otimes e^{-i\frac{\theta}{2}\vec{\sigma}_2\cdot\hat{n}}$$

Are there (nontrivial) invariant subspaces for $\rho_{\frac{1}{2}\otimes\frac{1}{2}}$

$$V_0 = \left\{ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \right\}$$ singlet state $J=0$

invariant

$$\langle v_0 | \rho_{\frac{1}{2}\otimes\frac{1}{2}}(\hat{n},\theta) | v_0 \rangle = 1$$

$$V_1 = \left\{ \underset{m=+1}{|\uparrow\uparrow\rangle},\ \underset{m=0}{\tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)},\ \underset{m=-1}{|\downarrow\downarrow\rangle} \right\}$$

$J=1$

$$\rho_{\frac{1}{2}\otimes\frac{1}{2}}(\hat{n},\theta) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & e^{-i\hat{n}\cdot\vec{L}\theta} \end{array}\right)$$

L - spin 1 matrices

$$\rho_{\frac{1}{2}\otimes\frac{1}{2}} \cong \rho_0 \oplus \rho_1 \qquad \left(\tfrac{1}{2}\otimes\tfrac{1}{2} = 0 \oplus 1\right)$$

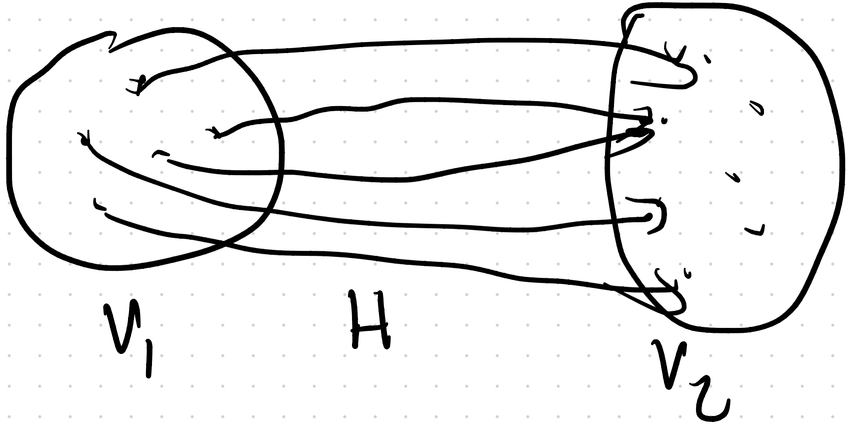
Clebsch-Gordan coefficients : matrix elements of the change of basis that block-diagonalises ρ

Schur's Lemma (2 ½ parts)

Part I: Let G be a group

$\rho_1 : G \to O(V_1)$ be two irreducible representations
$\rho_2 : G \to O(V_2)$

and let $H : V_1 \to V_2$ be a linear map (rectangular matrix)



Such that

$$He_1(g) = e_2(g)H$$

for all $g \in G$

Then either ① $H = 0$

or

② H is invertible

Proof: if $H = 0$: $\ker H = V_1$

$\operatorname{Im} H = \{\vec{0}\}$

if H is invertible $\operatorname{Im} H = V_2$

$\ker H = \{\vec{0}\}$

Look first @ $\ker H = \{ |v_1\rangle \in V_1 \mid H|v_1\rangle = \vec{0} \}$

$|v_1\rangle \in \ker H$ $\quad H\rho_1(g)|v_1\rangle = \rho_2(g)\underbrace{H|v_1\rangle} = 0$

$\Rightarrow \rho_1(g)|v_1\rangle \in \ker H$

$\ker H$ is an invariant subspace for the irreducible representation ρ_1

$$\Rightarrow \ker H = \begin{cases} \{\vec{0}\} \\ V_1 \to H = 0 \end{cases}$$

$$\boxed{H\rho_1(g) = \rho_2(g) H}$$

Now $\mathrm{Im}\, H = \{ |v_2\rangle \in V_2 \mid |v_2\rangle = H|v_1\rangle \text{ for some } |v_1\rangle \in V_1 \}$

$|w\rangle \in \mathrm{Im}\, H \Rightarrow |w\rangle = H|v\rangle \quad |v\rangle \in V_1$

$\rho_2(g)|w\rangle = \rho_2(g) H|v\rangle = H \rho_1(g)|v\rangle$

$\Rightarrow \rho_2(g)|w\rangle \in \mathrm{Im}\, H \rightarrow \mathrm{Im}\, H$ is an invariant subspace for ρ_2

$$\mathrm{Im}\, H = \begin{cases} V_2 \rightarrow H \text{ is invertible} \leftarrow \{\vec{0}\} \\ \{\vec{0}\} \rightarrow H = 0 \leftarrow V_1 \end{cases} \Bigg\} = \mathrm{Ker}\, H$$

Part 2

$V_1 = V_2 = V$ and $\rho_1 = \rho_2 = \rho$ irreducible

V is finite dimensional and suppose H satisfies

$$H\rho(g) = \rho(g)H \rightarrow [H, \rho(g)] = 0$$

for all $g \in G$

then either ① $H = 0$

or

② $H = \lambda\, Id_V$ identity matrix

Pf: Part 1 $\Rightarrow$ $H = 0$ or H is invertible

so assume H is invertible

$\Rightarrow$ H has at least one eigenvector $|v\rangle$

$$H|v\rangle = \lambda|v\rangle$$

$B = H - \lambda \mathrm{Id}_V$ $\qquad [B, \rho(g)] = 0$

$\Rightarrow$ Part 1 says $B = 0$ or B is invertible

but B can't be invertible b/c $\det B = 0$

$\Rightarrow B = 0 \Rightarrow H = \lambda \mathrm{Id}_V$

Part 2.5) G is a group

$\rho_1 : G \to U(V_1)$ irreducible, finite dimensional

$\rho_2 : G \to U(V_2)$ $\qquad H : V_1 \to V_2$

$H \rho_1(g) H^{-1} = \rho_2(g)$

$H \rho_1(g) = \rho_2(g) H$ forall $g \in G$ $\qquad H \neq 0$

$\rightarrow$ ρ_1 & ρ_2 are (unitarily) equivalent

Pf: Consider $H^\dagger : V_2 \to V_1$ $\qquad$ $V_1 \ni (\langle v_2|H)^\dagger = H^\dagger|v_2\rangle$

$H^\dagger \neq 0$

$H^\dagger(\rho_2(g))^\dagger = \rho_1(g)^\dagger H^\dagger$

$H^\dagger \rho_2(g^{-1}) = \rho_1(g^{-1}) H^\dagger$ forall $g^{-1} \in G$

$\rightarrow$ $H^\dagger$ also satisfies our assumptions for part 1

$\rightarrow$ $H^\dagger$ is invertible

$H^\dagger H : V_1 \to V_2 \to V_1$ $\qquad$ $[H^\dagger H, \rho_1(g)] = 0$

$H^\dagger H \neq 0$ $\quad$ Part 2 $\rightarrow$ $H^\dagger H = \lambda \, Id_{V_1}$

$$H^\dagger = \lambda H^{-1}$$

define $U = \frac{1}{\sqrt{\lambda}} H$

$$U^\dagger = \frac{1}{\sqrt{\lambda}} H^\dagger = \sqrt{\lambda} H^{-1} = U^{-1}$$

$$U \rho_1(g) = \rho_2(g) U$$

$$\boxed{U \rho_1(g) U^\dagger = \rho_2(g)} \rightarrow \rho_1 \cong \rho_2$$