Lecture 3

Announcements:
- HW 1 will be posted on Thursday (due 9/18)
- Office Hrs start next week Tuesdays 2-3pm (Zoom link on course website)

Recap: Normal subgroup $H \trianglelefteq G$

H is a subgroup and $gH = Hg$

$gHg^{-1} = H$ for all $g \in G$

G/H quotient group

Group Homomorphism $\phi : G \to K$

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$$

$K \supseteq \text{Im}\, \phi = \{ \varphi(g) \mid g \in G \}$

$G \supseteq \text{Ker}\, \varphi = \{ g \in G \mid \varphi(g) = E_K \}$

① $\text{Im}\, \phi \leq K$ is a subgroup of K

ⓐ $E_K = \phi(E_G)$ $E_K \in \text{Im}\, \phi$

$$\begin{aligned} \phi(E_G) &= \phi(g \cdot g^{-1}) \\ &= \phi(g) \cdot \phi(g^{-1}) \\ &= \phi(g) \cdot [\phi(g)]^{-1} \\ &= E_K \end{aligned}$$

ⓑ $k_1, k_2 \in \text{Im}\, \phi$

$k_1 = \phi(g_1)$, $k_2 = \phi(g_2)$

$$k_1 \cdot k_2 = \phi(g_1) \cdot \phi(g_2) = \phi(g_1 \cdot g_2) \in \operatorname{Im} \phi$$

c) $k_1 \in \operatorname{Im} \phi$

$$k_1 = \phi(g_1) \Rightarrow k_1^{-1} = [\phi(g_1)]^{-1} = \phi(g_1^{-1}) \in \operatorname{Im} \phi$$

② $\operatorname{Ker} \phi \trianglelefteq G$ is a normal subgroup of G

a) $\phi(E_G) = E_K \Rightarrow E_G \in \operatorname{Ker} \phi$

b) $g_1, g_2 \in \operatorname{Ker} \phi \quad \phi(g_1) = E_K, \ \phi(g_2) = E_K$

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2) = E_K \cdot E_K = E_K$$

$\Rightarrow g_1 \cdot g_2 \in \operatorname{Ker} \varphi$

c) $g_1 \in \ker\varphi$ $\quad$ $\underset{\phi(E_G) = E_K}{\underset{\parallel}{\phi(g_1)}} \cdot \phi(g_1^{-1}) = \phi(g_1^{-1})$

$g_1^{-1} \in \ker\varphi$

d $\quad g \in \ker\varphi \quad g' \in G$

$g' g g'^{-1} \overset{?}{\in} \ker\varphi$

$$\varphi(g' g g'^{-1}) = \varphi(g') \cdot \overset{E_K}{\cancel{\varphi(g)}} \cdot \varphi(g'^{-1})$$

$$= \varphi(g') \cdot \varphi(g'^{-1}) = \varphi(g' g'^{-1}) = E_K$$

$g' \ker\phi\, g'^{-1} = \ker\phi \rightarrow \ker\varphi \trianglelefteq G$

First isomorphism theorem: let G, K be groups, $\varphi: G \to K$ a group homomorphism

$G/\ker\varphi$ is isomorphic to $\mathrm{Im}\,\varphi$

there is an invertible homomorphism

$$\phi(g^{-1}) = [\phi(g)]^{-1}$$

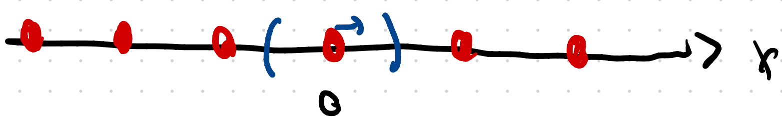
$$\phi(E_G) = E_K$$

$$[\phi(g)]^{-1}\cancel{\phi(g)} = E_K$$

$$= \phi(g^{-1})\cancel{\phi(g)}$$

Example: $\mathbb{R}^1 = \{\beta a\hat{x} \mid \beta \in \mathbb{R}\}$

$$T = \{n a \hat{x} \mid n \in \mathbb{Z}\}$$



$\gamma \in (-\frac{1}{2}, \frac{1}{2}]$

Claim: $\mathbb{R}^1/T \cong U(1)$ the unit circle

Pf: look for a homomorphism w/ $\text{Im} = U(1)$, $\text{Ker} = T$

$$\varphi(\beta a\hat{x}) = e^{i2\pi\beta}$$

is it a homomorphism $\varphi(\beta_1 a\hat{x} + \beta_2 a\hat{x}) = e^{2\pi i\beta_1} e^{2\pi i\beta_2} = \varphi(\beta_1 a\hat{x})\varphi(\beta_2 a\hat{x})$

✓✓

$$\text{Ker}\,\varphi : \{\beta a\hat{x} \mid e^{2\pi i\beta} = 1\} = \{n a\hat{x} \mid n \in \mathbb{Z}\} = T$$

$\text{Im}\,\varphi$: entire unit circle

First isomorphism theorem $\mathbb{R}^1/T \cong U(1)$ the unit cell of the Bravais lattice

One last thing about quotient groups:

let $H \triangleleft G$ be a normal subgroup

$$G = H \cup Hg_1 \cup \dots \cup Hg_{n-1} \qquad n = |G:H|$$

$G/H = \{H, Hg_1, \dots, Hg_{n-1}\}$ is a group

In <u>some</u> cases we can identify elements of G/H to elements of G

$$i : G/H \to G \qquad \text{"inclusion"}$$

$$i(Hg_i) = \hat{g}_i \in Hg_i$$

$$i(H) = E$$

if i exists, $\mathrm{Im}\, i = \{E, \hat{g}_1, \hat{g}_2, \dots \hat{g}_{n-1}\} \leq G$

$$G/H \cong \mathrm{Im}\, i$$

$$G = \bigcup_{i=0}^{n-1} H \bar{g}_i$$

every element of G can be written uniquely as $g = hk$ $\quad h \in H$, $k \in \mathrm{Im}\, i$

$$G = H(\mathrm{Im}\, i)$$

if this is possible G is a semidirect product

$$G = H \rtimes \mathrm{Im}\, i$$

Example: Rigid transformations of 3D space

Euclidean group $\mathbb{E}(3)$ - translations

- rotations
- reflections

elements $g \in \mathbb{E}(3)$ using a "Seitz symbol"

$$g = \{R \mid \vec{v}\} \qquad R \in O(3) \text{ rotation or reflection}$$

$\vec{v} \in \mathbb{R}^3$ is a translation

$$\{R|\vec{v}\} \cdot \vec{x} = [R]\vec{x} + \vec{v}$$

$[R]$ = 3x3 matrix that implements R

multiplication: $g_1 = \{R_1|\vec{v}_1\} \quad g_2 = \{R_2|v_2\}$

$$g_1 \cdot g_2 \vec{x} = g_1 \cdot (g_2 \vec{x}) = g_1 \cdot ([R_2]\vec{x} + \vec{v}_2)$$

$$= [R_1] \cdot ([R_2]\vec{x} + v_2) + \vec{v}_1$$

$= [R_1 R_2]\vec{x} + (v_1 + [R_1]v_2)$

$g_1 \cdot g_2 = \{R_1 \cdot R_2 \mid v_1 + R_1 \vec{v}_2\}$

inverses: $\{R \mid v\}^{-1} = \{R^{-1} \mid -R^{-1}\vec{v}\}$

identity: $\{E \mid \vec{0}\}$

1. $O(3) < \mathbb{E}(3)$
   $\quad \shortparallel$
   $\{\{R \mid \vec{0}\},\ R \in O(3)\}$

2. $\mathbb{R}^3 = \{\{E \mid \vec{v}\} \mid v \in \mathbb{R}^3\} \triangleleft \mathbb{E}(3)$ is a normal subgroup

$(\{R \mid \vec{d}\}\{E \mid \vec{v}\})\{R \mid \vec{d}\}^{-1} = \{R \mid \vec{d} + R\vec{v}\}\{R^{-1} \mid -R^{-1}\vec{d}\}$

$= \{E \mid \vec{d} + R\vec{v} - \vec{d}\} = \{E \mid R\vec{v}\}$

$$\{R \mid \vec{v}\} = \{E \mid \vec{v}\}\{R \mid \vec{0}\}$$

$$\mathbb{E}(3) = \mathbb{R}^3 \rtimes O(3)$$

$$\mathbb{E}(3) = \bigcup_{R \in O(3)} (\mathbb{R}^3) R$$

$\mathbb{E}(3)$ is a semidirect product

V = Hilbert space of states $|\psi\rangle$

G group of symmetries of our Hamiltonian

$\rho : G \to U(V)$ ← group of unitary operators on V

$g \to \rho(g)$ $\quad \rho(g)|\psi\rangle = |\psi'\rangle$

$\rho(g)^\dagger \hat{O} \rho(g) = \hat{O}'$ $\quad$ give transformed states and operators

$\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ so ρ should be a homomorphism

<u>Def</u> a vector space V and a homomorphism $\rho : G \to U(V)$ is called a <u>(unitary) representation</u> of G on V

V - the representation space

$\rho(g)$ - the representative of g

Ex: representation of translations $\mathbb{R}^3$ on Hilbert space

$V = \{\psi(\vec{x})$ square-integrable wavefunctions$\}$

$$\mathbb{R}^3 \ni \vec{v} \longrightarrow \rho(\vec{v}) = e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{v}} \in U(V)$$

$$\rho(v_1 + v_2) = \rho(v_1)\rho(v_2)$$

$$\rho(\vec{v})^\dagger \vec{x}\, \rho(v) = e^{\frac{i}{\hbar}\vec{p}\cdot v}\, \hat{x}\, e^{-\frac{i}{\hbar}p\cdot v} = \hat{x} + \vec{v}$$

More complicated example: $SU(2)$ ← special unitary group in 2D

$$= \{(\hat{n}, \theta) \mid \theta \in (-2\pi, 2\pi)\}$$

$\hat{n}$: 3D unit vector

$$(\hat{n}, \theta = 0) = E$$

$$(\hat{n}, \theta = 2\pi) \doteq \bar{E}$$

spin-$\frac{1}{2}$ representation: $\rho_{\frac{1}{2}}((\hat{n}, \theta)) = \cos\frac{\theta}{2}\,\sigma_0 + i\sin\frac{\theta}{2}\,\hat{n}\cdot\vec{\sigma}$

σ_0: 2×2 identity

$\vec{\sigma}$: vector of pauli matrices

$$\parallel$$

$$e^{-i\frac{\theta}{2}\hat{n}\cdot\sigma}$$

<u>defining representation</u>

$$\mathrm{Ker}\,\rho_{\frac{1}{2}} = \{E\}$$

$$\mathrm{Im}\,\rho_{\frac{1}{2}} = SU(2)$$

Other representations: spin $\ell = 1$

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \rho_1((\hat{n}, \theta)) = e^{-i\hat{n}\cdot\vec{L}\theta}$$

$$L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$\operatorname{Ker} \rho_1 = \{E, \bar{E}\}$$

"2π rotation"

$$\operatorname{Im} \rho_1 = SO(3) = SU(2) / \{E, \bar{E}\}$$

"SU(2) is a double cover SO(3))