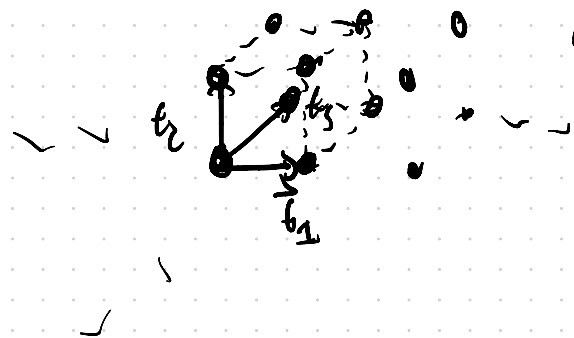
Lecture 2

Recap: group G - set with:

- associative "multiplication
- identity element
- every element has an inverse

Ex: Bravais lattice



$$T = \left\{ \sum_{i=1}^{3} n_i \vec{t}_i \;\middle|\; n_i \in \mathbb{Z},\ t_i \text{ linearly independent} \right\}$$

binary operation: $+$

identity: $\vec{0}$

$$\left(\sum n_i \vec{t}_i\right)^{-1} = \sum -n_i \vec{t}_i$$

Subgroup $H \leq G$ if $H \subseteq G$ and H is also a group w/ the same binary operation

(Note: Every group G has at least 2 subgroups

- $G \leq G$ - the whole group
- $\{E\} \leq G$ trivial group)

Right cosets: $Hg_i = \{hg_i \mid h \in H\}$

partition the group $G = \bigcup_{i=0}^{n-1} Hg_i$ n = # of cosets

index of H in G

$|G:H|$

Define: conjugation by $g_i \in G$

$$C_{g_i}(g) \to g_i g g_i^{-1}$$

Conjugate a subgroup H by g_i

$$\{ g_i h g_i^{-1} \mid h \in H \} = g_i H g_i^{-1}$$

Claim: if $H \leq G$ then $g_i H g_i^{-1} \leq G$

Pf $\cdot\ E \in H \to g_i E g_i^{-1} = E \qquad E \in g_i H g_i^{-1}$

$\cdot\ (g_i h_1 g_i^{-1}) \cdot (g_i h_2 g_i^{-1}) = g_i (h_1 h_2) g_i^{-1} \in g_i H g_i^{-1}$

$\cdot\ g_i h^{-1} g_i^{-1} \cdot g_i h g_i^{-1} = E$

So $g_i H g_i^{-1}$ is a group if H is a group

conjugate subgroup to H

Two elements $g_1 \in G$ and $g_2 \in G$ are <u>conjugate</u> if there's a $g_j \in G$ s.t.

$g_2 = g_i g_1 g_i^{-1}$

conjugacy class of g

$-\mathcal{C}(g) = \{ g_i g g_i^{-1} \mid \text{for all } g_i \in G \}$

if a is conjugate to b

and b is conjugate to c

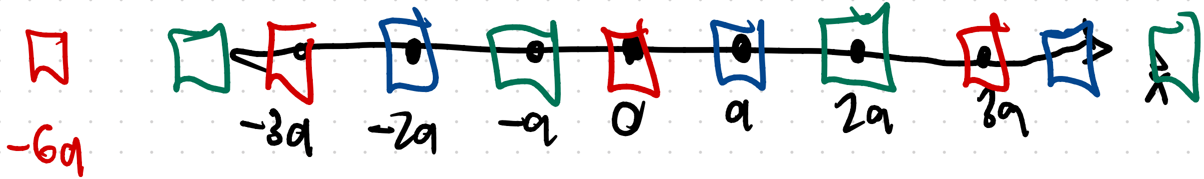
$$\rightarrow \quad a = g_i b g_i^{-1}$$
$$c = g_j b g_j^{-1} \quad \rightarrow a = (g_i g_j^{-1}) c (g_j g_i^{-1})$$

a is conjugate to c

$\rightarrow$ Conjugacy classes partition the group

<u>Define</u> $H \leq G$ is a <u>normal subgroup</u> if $gHg^{-1} = H$ for any $g \in G$ (H is invariant under conjugation)

Example (trivial) 1D Bravais lattice $T = \{ na\hat{x} \mid n \in \mathbb{Z} \}$
$< \mathbb{R}^1$



$H = \{3an\hat{x} \mid n \in \mathbb{Z}\}$

$H \leq T$

Right cosets:

$H = \{0, \pm 3a\hat{x}, \pm 6a\hat{x}, \ldots\}$

$H + a\hat{x} = \{a\hat{x}, 4a\hat{x}, 7a\hat{x}, \ldots, -2a\hat{x}, \ldots\}$

$H + 2a\hat{x} = \{2a\hat{x}, 5a\hat{x}, \ldots, -a\hat{x}, \ldots$

$T = H \cup [H + a\hat{x}] \cup [H + 2a\hat{x}]$

Conjugate H by $na\hat{x}$

$$n a\hat{x} + H + (-n a\hat{x}) = H$$

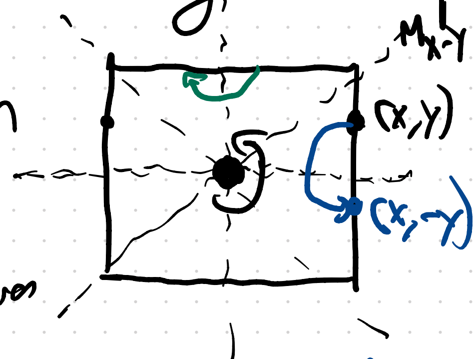
H is a normal subgroup of T

whenever our group operation is commutative ← abelian groups

every subgroup is normal $\quad g h g^{-1} = h g g^{-1} = h$

Slightly less easy example:

Rigid symmetries of a square



$E = 0°$ rotation

$C_4 = 90°$ rotation

$C_2 = C_4^2 = 180°$ rotation

$C_4^{-1} = C_4^3 = 270°$ rotation

$$\{E, C_4, C_2, C_4^{-1}, M_y, M_x, M_{x-y}, M_{x+y}\} = 4mm$$

M_y: mirror reflection $y \to -y$

$M_x : x \to y$ $\quad C_2 \cdot M_y = M_x$

Conjugation: $C_4 M_x C_4^{-1} = M_y$

Subgroup $H = \{E, M_x\}$

$$C_4 H C_4^{-1} = \{E, M_y\} \neq H$$

H is not a normal subgroup

$2mm = \{E, M_x, M_y, C_2\} \subset 4mm$ <u>is</u> a normal subgroup

symmetries of a rectangle

$$\left.\begin{array}{l} \{M_x, M_y\} \\ \{M_{x+y}, M_{x-y}\} \\ \{E\} \\ \{C_2\} \\ \{C_4, C_4^{-1}\} \end{array}\right\} \text{Conjugacy classes}$$

$$\underbrace{H \trianglelefteq G}_{\text{normal subgroup}}$$

For normal subgroups $H \trianglelefteq G$

then we can multiply Hg_1 and Hg_2

$$Hg_1 \cdot Hg_2 = \{hg_1 \cdot h'g_2 \mid h, h' \in H\}$$

H normal: $g_1 H g_1^{-1} = H$

$$g_1 h' g_1^{-1} = h'' \in H$$

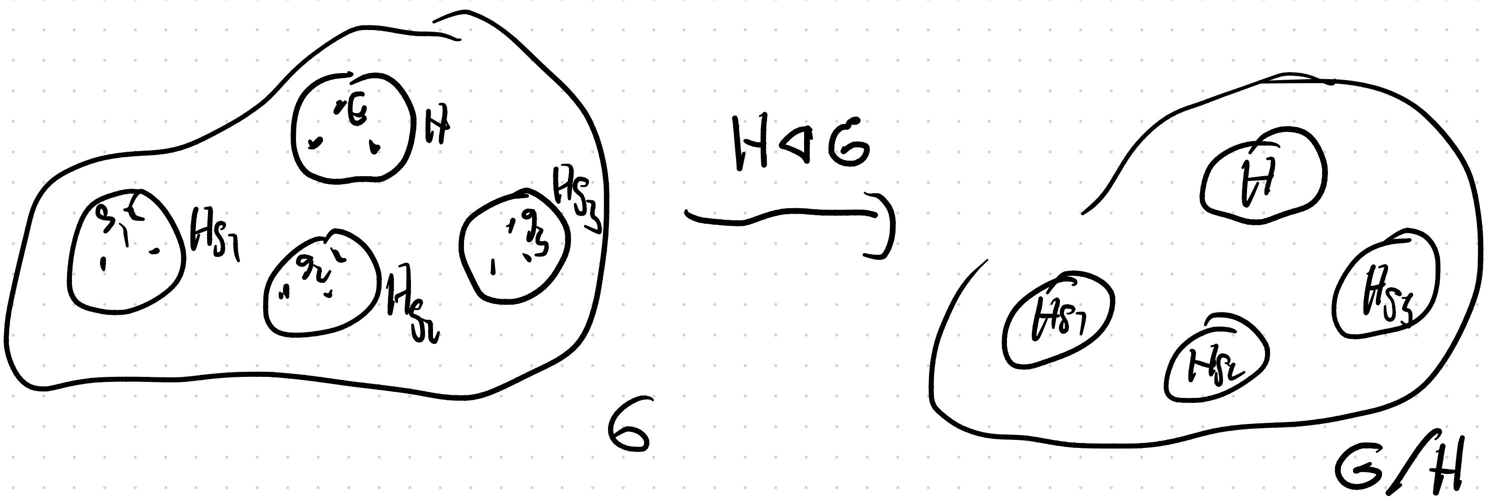
$$g_1 h' = h'' g_1$$

$$= \{h \cdot h'' g_1 g_2 \mid h, h'' \in H\}$$

$$= \{h \cdot g_1 g_2 \mid h \in H\} = Hg_1 g_2$$

$HE \cdot Hg = Hg$ $\qquad$ $(Hg) \cdot (Hg^{-1}) = H$

if $H \trianglelefteq G$ is a normal subgroup, the set $\{Hg \mid g \in H\}$ of right cosets of H forms a group - quotient group G/H



Ex: 1D Bravais lattice $T = \{n a\hat{x} \mid n \in \mathbb{Z}\}$

$H = \{3 n a\hat{x} \mid n \in \mathbb{Z}\}$

$[H] \sim [0]$

$[H + a\hat{x}] \sim [1]$

$[H + 2a\hat{x}] \sim [2]$

$[1] + [1] = [2]$

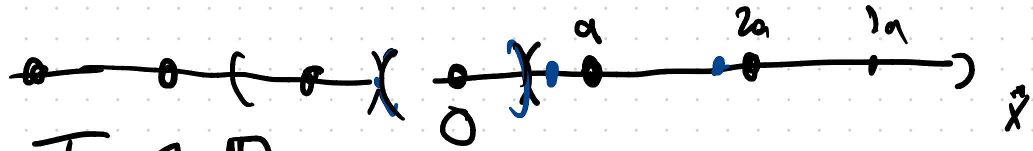
$[2] + [2] = [1]$

$[1] + [2] = [0]$

$(H + 2a\hat{x}) + (H + 2a\hat{x}) = H + a\hat{x} + 3a\hat{x}$

$T/H \cong \mathbb{Z}_3$ addition of integers mod 3

Ex: $\mathbb{R}$



$T \triangleleft \mathbb{R}$

$T = \{ n a \hat{x} \mid n \in \mathbb{Z} \}$

$\mathbb{R} = \{ \beta a \hat{x} \mid \beta \in \mathbb{R} \}$

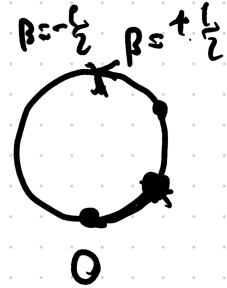
Q: $\mathbb{R}/T$

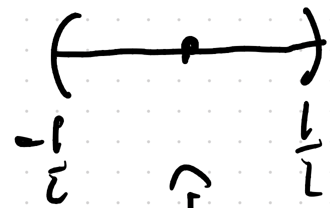
Right cosets $\quad T + \beta a \hat{x} \quad \beta \in [-\frac{1}{2}, \frac{1}{2})$

index $|\mathbb{R} : T| = \infty$

Quotient group $\quad [T + \beta a \hat{x}] \sim [\beta]$

$$[\beta_1] + [\beta_2] = T + (\beta_1 + \beta_2) a \hat{x}$$





Primitive unit cell for the Bravais lattice

In quantum mechanics

$\Phi : G \longrightarrow K$

group of symmetries

unitary operators on Hilbert space

$\phi: g \to \phi(g) \in K$

subset of ϕ compatible w/ group multiplication

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \leftarrow \underline{\text{group homomorphism}}$$

Ex: $T = \{\sum_i n_i \vec{t}_i \mid n_j \in \mathbb{Z}\}$

$$\phi(\sum_i n_i t_i) = e^{-\frac{i}{\hbar}\vec{p}\cdot(\sum_i n_i \vec{t}_i)} \in K \qquad \vec{p} = -i\hbar\frac{\partial}{\partial \vec{x}}$$ momentum operator

Given a group homomorphism $\phi: G \to K$

$E_G \in G$ identity in G

$E_K \in K$ identity in K

$$\phi(E_G) = E_K$$

Two subsets:

$\mathrm{Im}\,\varphi$: $\{\varphi(g) \mid g \in G\} \subset K$ image of φ

$\mathrm{Ker}\,\varphi$: $\{g \in G \mid \varphi(g) \in E_K\} \subset G$ kernel of φ