

Lecture 2:

Review: Last class we concluded that we must associate with each event in the quantum world a complex state vector ψ . Today we will clarify the meaning of this and delineate the properties of the space in which ψ sits.

1.) Vector spaces

In Q.M., the state of a system is uniquely specified by the complex state vector $|\psi\rangle$.

$|\psi\rangle$ is called a ket and $\langle\psi|$ a bra.

$|\psi\rangle$ resides in the space S , while $\langle\psi|$ sits in S^* . What is S .

S is a linear vector space of objects $|1\rangle, \dots, |n\rangle$. The number of linearly-independent vectors

in S determines its dimensionality. In S there is a definite rule addition and multiplication by scalars.

$$|S_1\rangle + |S_2\rangle \in S$$

$$a|S_1\rangle + b|S_2\rangle = a(|S_1\rangle + |S_2\rangle)$$

Vectors in the vector space are linearly independent if any linear combination of the vectors does not generate another member in the set.

(2)

Consider the example the vector space of real 2×2 matrices.

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, |3\rangle = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$$

Is this set linearly independent?

$$\text{Clearly } |3\rangle = -2|2\rangle + 3|1\rangle.$$

What is the dimensionality of the real vector space of 2×2 matrices? Consider:

$$|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, |3\rangle = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, |4\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Any 2×2 matrix can be written as a linear combination of these four matrices. \Rightarrow the dimensionality of the space is 4. Consider the hermitian conjugate space.

$$\langle 1 | = [|1\rangle^T]^* = |1\rangle^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\langle 2 | = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = |2\rangle^T, \langle 3 | = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = |3\rangle^T$$

$$\langle 4 | = |4\rangle^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = |4\rangle^T$$

(3.)

Vector Operations

a.) Norm

$$\|\phi\| = \sqrt{\langle \phi | \phi \rangle} \quad \text{length of vector}$$

If $\|\phi\| = 1$, then $|\phi\rangle$ has unit length.

b.) Inner product.

$$\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$$

c.) Orthogonality:

$$\langle \phi | \psi \rangle = 0.$$

Example: 2×2 matrices

$$\langle 2 | 1 \rangle = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \langle 2 | 1 \rangle$$

$\Rightarrow |1\rangle$ and $|2\rangle$ are not orthogonal.

In general if

$$|V\rangle = \sum_i v_i |i\rangle$$

$$|W\rangle = \sum_j w_j |j\rangle$$

$$\Rightarrow \langle V | W \rangle = \sum_{ij} v_i^* w_j \langle j | i \rangle.$$

(4.)

d.) Orthonormality:

Consider a set of vectors discretely indexed,

$$\langle \hat{e}_i | \hat{e}_j \rangle = S_{ij} \Rightarrow \text{orthonormality.}$$

$$= \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

S_{ij} is the Kronecker δ -function.

Consider continuously indexed functions.

$$\langle \psi_\alpha | \psi_{\alpha'} \rangle = \delta(\alpha - \alpha').$$

Example: plane waves

$$|\phi_k\rangle = \frac{e^{ik \cdot x}}{\sqrt{2\pi}} \quad k \in \mathbb{R}.$$

$$\langle \phi_{k'} | \phi_{k'} \rangle = \frac{1}{2\pi} \int dx e^{i(k' - k) x} = \delta(k - k').$$

The properties of the δ -fn. are as follows.

$$1.) \int f(x) \delta(x - x') dx = f(x').$$

$$\Rightarrow f(x) \delta(x - x') = f(x') \delta(x - x')$$

where $f(x)$ is a continuous fn. of $x \Rightarrow$

(5)

$$\int \delta(x) = 0.$$

$$2.) \quad \delta(x) = \delta(-x)$$

$$3.) \quad \delta(ax) = \frac{1}{|a|} \delta(x), \quad a > 0$$

2.) Superposition Principle: 2-slit experiment

Consider a state vector that is a L.C. (linear combination) of two pure states,

$$|\psi\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle.$$

$$\text{Let's assume } \langle \phi_i | \phi_j \rangle = \delta_{ij}.$$

What are the expansion coefficients c_i ?

$$\langle \phi_i | \psi \rangle = c_1 \langle \phi_1 | \phi_i \rangle + 0.$$

$$\Rightarrow c_1 = \langle \phi_1 | \psi \rangle.$$

$$\text{in general } c_i = \langle \phi_i | \psi \rangle.$$

$$\Rightarrow |\psi\rangle = |\phi_1\rangle \langle \phi_1 | \psi \rangle + |\phi_2\rangle \langle \phi_2 | \psi \rangle.$$

$$\Rightarrow |\phi_1\rangle \langle \phi_1 | + |\phi_2\rangle \langle \phi_2 | = 1.$$

This is a general result.

$$\sum_i |\phi_i\rangle \langle \phi_i | = 1 \quad \text{where } |\phi_i\rangle$$

(6)

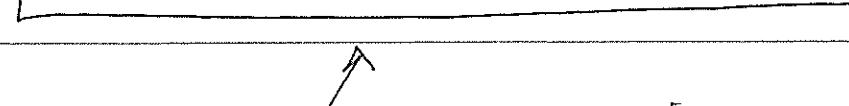
are a set of linearly independent vectors that span the vector space. This is sometimes called the spectral resolution of the identity. In our 2-slit experiment we only have two states $|\phi_1\rangle, |\phi_2\rangle$.

The total probability is

$$\begin{aligned}\langle \phi | \phi \rangle &= |\langle \phi | \phi_1 \rangle|^2 + |\langle \phi | \phi_2 \rangle|^2 \\ &= c_1^2 + c_2^2 = 1.\end{aligned}$$

This is our initial state. Now we measure the electron at a detector at x . Let $|\psi_x\rangle$ be the final state of the system. The joint probability of detecting the particle at x given that it is initially in $|\phi\rangle$ is

$$\begin{aligned}P_x &= |\langle \psi_x | \phi \rangle|^2 \\ &= |\langle \psi_x | \phi_1 \rangle c_1 + \langle \psi_x | \phi_2 \rangle c_2|^2 \\ &= |c_1|^2 |\langle \psi_x | \phi_1 \rangle|^2 + |c_2|^2 |\langle \psi_x | \phi_2 \rangle|^2 \\ &\quad + c_1 c_2^* \langle \psi_x | \phi_1 \rangle \langle \phi_2 | \psi_x \rangle + h.c.\end{aligned}$$

interference terms.

(7)

There are three contributions: a) P_1 , b) P_2 and
 c) the interference arising from the cross terms.
 The fact that that $\langle \Psi_1 | \Psi_2 \rangle = 0 \Rightarrow$
 $\langle \Psi_2 | \Psi_1 \rangle$ overlap with both $|\Psi_1\rangle$ and $|\Psi_2\rangle$
 at the same time. This is not true in the
 bullet problem.

Now let's ask the question: What is
 the problem that an electron passes through
 slit 1.

$$P_1 = |\langle \Psi_1 | \Psi \rangle|^2 = |C_1|^2$$

Technically, we must divide by $|C_1|^2 + |C_2|^2$.

$$\Rightarrow P_1 = \frac{|C_1|^2}{|C_1|^2 + |C_2|^2}$$

$$\text{but } |C_1|^2 + |C_2|^2 = 1$$

$|C_1|^2$ is the probability we would obtain
 if we were to shine a light at each slit
 and measure which slit the electron went
 through. \Rightarrow measurement collapses the
 state of the system. \Rightarrow measurement
 destroys interference.

Measurement Hypothesis: Consider two
 pure states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.

Assume there exists an observation of some variable in $|X_1\rangle$ that yields the definite result a_1 and a_2 in $|X_2\rangle$. Consider now the mixed state $|U\rangle = C_1|X_1\rangle + C_2|X_2\rangle$.

Measurement of the observable in $|U\rangle$ yields either a_1 with probability $|C_1|^2$ or a_2 with probability $|C_2|^2$. The uncertainty can't be removed until a measurement is made. This of course leads to peculiar circumstances for example in dead + live cat paradox.

$$|U_{\text{mixed}}\rangle = C_1(\text{dead cat}) + C_2(\text{alive cat}).$$

3.) Linear Operators:

An operator is an instruction for transforming any given vector into another one.

$$\hat{O}|X\rangle = |X'\rangle$$

\hat{O} can also act on a bra in the dual space S^*

$$\langle X|\hat{O} = \langle X'|.$$

We are concerned with linear

(9)

linear operators

$$\hat{O}c|x\rangle = c\hat{O}|x\rangle = c|x\rangle$$

$$\Rightarrow \text{if } |k\rangle = \sum_i c_i |x_i\rangle.$$

$$\Rightarrow \hat{O}|k\rangle = \sum_i c_i |\hat{x}_i\rangle \text{ by linearity.}$$

In some instances

$$\hat{O}|x\rangle = \lambda|x\rangle \text{ when } \lambda \text{ is a number.}$$

In such instances; \hat{O} is an eigenoperator with eigenfunction $|x\rangle$ and λ is the eigenvalue.

Postulate: For every observable in Q.M. there exists an associated Hermitian operator: $\hat{O} = \hat{O}^+$. \hat{O}^+ is the adjoint of \hat{O} .

Hermitian operators have the following properties.

a.) real eigenvalues

$$\text{Proof: } \langle x|\hat{O}|x\rangle = \lambda \langle x|x\rangle.$$

$$\text{Consider } \langle x|\hat{O}^+|x\rangle$$

$$\text{note } \langle x|\hat{O}^+ = [\hat{O}|x\rangle]^* = \langle x|\lambda^*$$

$$\Rightarrow \langle x|\hat{O}|x\rangle = \lambda = \lambda^* \Rightarrow \lambda \text{ must be real.}$$

(10)

~~or~~ 2.) orthogonality of eigenvectors.

$$\hat{O}|x\rangle = \lambda|x\rangle, \quad \hat{O}|x'\rangle = \lambda'|x'\rangle.$$

$$\Rightarrow \langle x|x'\rangle = 0.$$

Consider

$$\begin{aligned} \langle x'|\hat{O}|x\rangle &= \lambda \langle x'|x\rangle \\ &= \lambda' \langle x'|x\rangle \end{aligned}$$

\Rightarrow either $\lambda = \lambda'$ or $\langle x'|x\rangle = 0$.

if $|x\rangle$ and $|x'\rangle$ are non-degenerate \Rightarrow
 $\langle x'|x\rangle = 0$.

3.) Linearity:

$$\hat{O}[|x\rangle + |x'\rangle] = \lambda|x\rangle + \lambda'|x'\rangle$$

4.) Non-commutativity.

$$\hat{O}_1 \hat{O}_2 |x\rangle \neq \hat{O}_2 \hat{O}_1 |x\rangle$$

$$\Rightarrow [\hat{O}_1, \hat{O}_2] - [\hat{O}_2, \hat{O}_1] \neq 0.$$

$[\hat{O}_1, \hat{O}_2]$ is a commutator.

Lecture 3:

1.) More on Operators

a.) commutators

What is meant by the operator product

$$\hat{O}_1 \hat{O}_2 |V\rangle ?$$

where $|V\rangle$ is some vector in the space \mathbb{V}

We proceed in sequence since they are linear op.

$$\hat{O}_1 (\hat{O}_2 |V\rangle) = \hat{O}_1 |\underbrace{\hat{O}_2 V}\rangle$$

We call $\hat{O}_2 |V\rangle$ a new ket $|\hat{O}_2 V\rangle$.

$$\Rightarrow \hat{O}_1 \hat{O}_2 |V\rangle = |\hat{O}_1 \hat{O}_2 V\rangle.$$

What if we had switched the order?

$$\hat{O}_2 \hat{O}_1 |V\rangle = |\hat{O}_2 \hat{O}_1 V\rangle \neq |\hat{O}_1 \hat{O}_2 V\rangle.$$

The difference between these two is summed up in the commutator

$$[\hat{O}_1, \hat{O}_2] = \hat{O}_1 \hat{O}_2 - \hat{O}_2 \hat{O}_1$$

↑
Commutator.

Consider a simple case in which

(2)

$$J = R\left(\frac{\pi}{2}\hat{i}\right), \quad \hat{J} = R\left(\frac{\pi}{2}\hat{j}\right).$$



rotation by $\frac{\pi}{2}$ around
the unit vector \hat{i}



rotation around \hat{j}
by $\frac{\pi}{2}$.

$$\text{Let } |1\rangle = \hat{i}, \quad |2\rangle = \hat{j}, \quad |3\rangle = \hat{k}.$$

$$R\left(\frac{\pi}{2}\hat{i}\right) |1\rangle = |1\rangle. \quad R\left(\frac{\pi}{2}\hat{j}\right) |1\rangle = -|3\rangle$$

$$R\left(\frac{\pi}{2}\hat{i}\right) |2\rangle = |3\rangle. \quad R\left(\frac{\pi}{2}\hat{j}\right) |2\rangle = |2\rangle$$

$$R\left(\frac{\pi}{2}\hat{i}\right) |3\rangle = -|2\rangle. \quad R\left(\frac{\pi}{2}\hat{j}\right) |3\rangle = |1\rangle$$

\Rightarrow we can represent $R\left(\frac{\pi}{2}\hat{i}\right)$ in
matrix form. To do this we simply form
the matrix \hat{O}_{ij}

$$\hat{O}_{ij} = \langle i | \hat{O} | j \rangle \quad \text{there are 9 such elements.}$$

$$\hat{O} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

likewise $\hat{J} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

(3.)

$$\hat{O}\hat{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{\Omega}\hat{O} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

b.) Projection

For a complete basis, $|i\rangle$

$$\sum_i |i\rangle \langle i| = 1$$

$|i\rangle \langle i|$ is called a projection operator. What does this mean.

$$\text{Let } |V\rangle = \sum_j V_j |j\rangle$$

What is V_j ? multiplying by $\langle i|$

$$\langle i|V\rangle = \sum_j V_j \underbrace{\langle i|j\rangle}_{S_{ij}} = V_i$$

$$\Rightarrow |V\rangle = \sum_j |j\rangle \langle j|V\rangle = |V\rangle$$

(4)

Now multiply by P_i

$$P_i |V\rangle = \sum_j |i\rangle \underbrace{\langle i|}_{\delta_{ij}} \langle j|V\rangle \\ = V_i |i\rangle.$$

$\Rightarrow P_i$ projects out the state $|i\rangle$ from the linear superposition $\sum_i V_i |i\rangle$. It is for this reason that it is called a projection operator.
 $1 - P_i$ is the orthogonal projector.

c.) More on Adjoints:

The Hermitian Conjugate of an operator is \hat{O}^+ . How do we understand

$$\langle V | \hat{O}^+ = ?$$

Let us look at $\hat{O}^+ |V\rangle$.

$$\langle \hat{O}^+ V | = \langle \hat{O} V |$$

now let's pull out the operator

$$\Rightarrow \langle \hat{O} V | = \langle V | \hat{O}^+$$

$$\text{if } \hat{O} |V\rangle = \lambda |V\rangle \Rightarrow \langle V | \hat{O}^+ = \langle V | \lambda^*.$$

To get back to our original question, we note that we need an expression for $(\hat{O}^2)^+ = ?$

(5.)

Consider $\langle \hat{O} \hat{\Omega} V |$. Let's move $\hat{\Omega}$ outside

$$\Rightarrow \langle \hat{O} \hat{\Omega} V | = \langle V | (\hat{O} \hat{\Omega})^+$$

Let us treat $\langle (\hat{O} V) |$ as a vector.

$$\Rightarrow \langle \hat{O} \hat{\Omega} V | = \langle \hat{O} | (\hat{\Omega} V) | = \langle \hat{\Omega} V | \hat{\Omega}^+$$

$$\Rightarrow \langle \hat{\Omega} V | \hat{\Omega}^+ = \langle V | \hat{\Omega}^+ \hat{\Omega}^+$$

$$\Rightarrow (\hat{O} \hat{\Omega})^+ = \hat{\Omega}^+ \hat{\Omega}^+$$

$$\Rightarrow [\langle V | \hat{\Omega}]^+ = \hat{\Omega}^+ | V \rangle$$

For an antihermitian operator $\hat{\Omega}^+ = -\hat{\Omega}$.

d.) Eigenvalues:

The eigenvalue problem is

$$\hat{\Omega} | V \rangle = \lambda | V \rangle$$

Let us rewrite this as

$$(\hat{\Omega} - \lambda I) | V \rangle = | 0 \rangle.$$

↑ null vector

(a bunch of zero's)

This equation only has a solution

if

$$\det(\hat{\Omega} - \lambda I) = 0.$$

(6)

This is the eigenvalue equation. This is called the Characteristic Equation. What does this eq. mean? We simply project onto a basis and form

$$\sum_j (R_{ij} - \lambda S_{ij}) v_j = 0$$

Example

$$R(\frac{I+1}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad \text{note } R^+ \neq R.$$

$$(\hat{R} - \lambda I) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0-\lambda \end{pmatrix} = 0$$

$$\det(R - \lambda I) = 0 \Rightarrow (1-\lambda)(\lambda^2 + 1) = 0.$$

$$\Rightarrow \lambda = 1, \pm i.$$

find eigenvectors

$$\lambda = 1$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & x_1 \\ 0 & -1 & -1 & x_2 \\ 0 & 1 & -1 & x_3 \end{array} \right) = 0 \Rightarrow \begin{aligned} x_1 &= \text{arbitrary} \\ x_2 + x_3 &= 0 \Rightarrow x_2 = x_3 = 0. \\ x_2 - x_3 &= 0 \Rightarrow \end{aligned}$$

$$\lambda = i \quad \left(\begin{array}{ccc|c} 1-i & 0 & 0 & x_1 \\ 0 & -i & -1 & x_2 \\ 0 & 1 & -i & x_3 \end{array} \right) = 0 \quad \begin{aligned} x_1(1-i) &= 0 \\ -x_2i - x_3 &= 0 \Rightarrow x_2 = i \\ x_2 - x_3i &= 0 \quad x_3 = 1 - \end{aligned}$$

$$|\omega=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |\omega=+i\rangle = \begin{pmatrix} 0 \\ +i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

normalisation factor.

if $\hat{\Omega}$ is Hermitian \Rightarrow exists a unitary matrix U , $\Rightarrow U^\dagger \hat{\Omega} U$ is diagonal. A unitary matrix satisfies $U^\dagger U = 1$. U is formed from the eigenvectors of $\hat{\Omega}$. What about two operators that commute. Assume $[\hat{\Omega}, \hat{A}] = 0 \Rightarrow$ they share common eigenvectors. ~~all~~

Proof: Let $\hat{\Omega}|i\rangle = \lambda_i|i\rangle$.

$$\hat{A}\hat{\Omega}|i\rangle = \lambda_i\hat{A}|i\rangle.$$

$$\text{but } \hat{A}\hat{\Omega}|i\rangle = \hat{\Omega}\hat{A}|i\rangle = \lambda_i\hat{A}|i\rangle.$$

$\Rightarrow \hat{A}|i\rangle$ is an eigenvector of $\hat{\Omega}$ with eigenvalue λ_i . But because ~~the~~ ^{$\hat{\Omega}$ is} non-degenerate $\Rightarrow \hat{A}|i\rangle = \lambda_i|i\rangle \Rightarrow \hat{A}$ and $\hat{\Omega}$ share common eigenvectors.

2.) Continuous basis:

Let's extend the superposition principle

- to continuous bases: Rather than

$$\sum_i v_i|i\rangle$$

(8)

$$|X\rangle = \int d\alpha X(\alpha) |2\alpha\rangle.$$

continuous basis.

What is $X(\alpha)$? multiply by $|2\alpha\rangle^+$.

$$\Rightarrow X(\alpha) = \langle 2\alpha | X \rangle.$$

There are two continuous representations we will use: a) \vec{r} and b) \vec{P} .

a.) Position space.

Let us associate a ket $|\vec{r}'\rangle$ with a fcn. located at \vec{r}' :

$$\langle \vec{r} | \vec{r}' \rangle = \delta(\vec{r} - \vec{r}') = \phi_1(r)$$

We now define a state vector

$$|\psi\rangle = \int d^3r \psi(r) |\vec{r}\rangle$$

where $\boxed{\psi(r) = \langle \vec{r} | \psi \rangle}$

$\psi(r)$ will play the role of the wavefunction. This is the probability amplitude of finding ~~the~~ a particle at \vec{r} .

b.) momentum space:

$$\vec{P} = \hbar \vec{k}$$

(9.)

At the moment let's set $k=1$. A plane-wave state is given by

$$\phi_{\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{r}}$$

$$|\vec{k}\rangle = \int \frac{d^3 r}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{r}} |\vec{r}\rangle = \int d^3 r \phi_{\vec{k}}(\vec{r}) |\vec{r}\rangle$$

$$\langle \vec{r}' | \vec{k} \rangle = \int d^3 r \phi_{\vec{k}}(\vec{r}) \underbrace{\langle \vec{r}' | \vec{r} \rangle}_{\delta(\vec{r}-\vec{r}')} = \phi_{\vec{k}}(\vec{r}')$$

\Rightarrow the $|\vec{r}\rangle$ and $|\vec{k}\rangle$ bases are related by complex phase factors.

$$\Rightarrow |\vec{k}\rangle = \int d^3 r |\vec{r}\rangle \langle \vec{r} | \vec{k} \rangle = \int d^3 r \hat{P}_{\vec{r}} |\vec{k}\rangle$$

The basis states $|\vec{k}\rangle$ are orthonormal. Consider

Projection operator.

$$\langle \vec{r}' | \vec{k} \rangle = \int d^3 \vec{r} d^3 \vec{r}' \underbrace{\langle \vec{k} | \vec{r} \rangle}_{\delta(\vec{r}-\vec{r}')} \langle \vec{r} | \vec{r}' \rangle \langle \vec{r}' | \vec{k} \rangle$$

$$= \int d^3 r \langle \vec{k} | \vec{r} \rangle \langle \vec{r} | \vec{k} \rangle$$

$$= \int \frac{d^3 r}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} e^{-i\vec{k} \cdot \vec{r}} = \delta(\vec{r})$$

Analogously we can write

$$|\vec{r}\rangle = \int d^3 k |\vec{k}\rangle \langle \vec{k} | \vec{r} \rangle$$

(10)

\Rightarrow We can expand an arbitrary state vector $|\psi\rangle$ as in both momentum and position space as

$$|\psi\rangle = \int d^3\vec{r} \psi(\vec{r}) |\vec{r}\rangle$$

$$\Rightarrow \langle k | \psi \rangle = \psi(k) = \int d^3r \psi(r) \langle k | r \rangle$$

$$= \int \frac{d^3r}{(2\pi)^{3/2}} e^{-ik \cdot r} \psi(r)$$

similarly,

$$|\psi\rangle = \int d^3k \psi(k) |k\rangle$$

$$\Rightarrow \langle \vec{r} | \psi \rangle = \int d^3k \psi(k) \langle \vec{r} | k \rangle$$

$$= \int d^3k \psi(k) \frac{e^{ik \cdot r}}{(2\pi)^{3/2}}$$

Lecture 4:

1.) More on measurement:

Consider a state vector that is described by the general superposition,

$$|\psi\rangle = \sum_i \gamma_i |i\rangle.$$

Consider also a linear hermitian operator for which

$$\hat{O}|i\rangle = \alpha_i |i\rangle, \alpha_i \text{ are real}.$$

Let's assume the eigenstates $|i\rangle$ are non-degenerate. What do we obtain by applying \hat{O} to $|\psi\rangle$?

$$\hat{O}|\psi\rangle = \sum_i \gamma_i \alpha_i |i\rangle \neq (\text{number}) |\psi\rangle.$$

$\Rightarrow |\psi\rangle$ is not an eigenstate of \hat{O} .

So does this mean the system is not in a definite state? No. When we perform a measurement, we project out the state in which the system is.

Let us define the expectation value of the operator \hat{O} .

$$\langle \hat{O} \rangle = \frac{\langle \psi | \hat{O} | \psi \rangle}{\langle \psi | \psi \rangle} = \sum_i |\gamma_i|^2 \alpha_i$$

\Rightarrow each eigenvalue is weighted with

(2)

The probability $|c_i|^2 = |\langle i|\psi \rangle|^2$. Measurement is equivalent to applying the projector $P_i = |i\rangle\langle i|$ to $|\psi\rangle$.

$$|\psi\rangle \xrightarrow[\alpha_i]{\hat{O}} P_i |\psi\rangle$$

this is the new state generated after measurement.

We can normalise it appropriately.

$$\Rightarrow |\psi\rangle \xrightarrow{\hat{O}} \frac{P_i |\psi\rangle}{\|P_i |\psi\rangle\|} = \frac{|i\rangle \langle i| \psi\rangle}{\sqrt{|\langle i|\psi \rangle|}}$$

\Rightarrow implies that after measurement, all that is left of the original state is the phase factor

$$c^i = \frac{\langle i|\psi \rangle}{\sqrt{|\langle i|\psi \rangle|}}$$

\Rightarrow measurement creates the state $|i\rangle e^{i\theta}$.

2.) Uncertainty Principle

We now want to focus on the limitation on measuring two observables simultaneously.

(3.)

For any operator \hat{A} , the root-mean-square (RMS) deviation is

$$\Delta A = \sqrt{\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle}$$

$$= \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}.$$

A state with a definite value of A has $\Delta \hat{A} = 0$.

We want to show that for any two operators \hat{A} and \hat{B}

$$\Delta \hat{A} \Delta \hat{B} = \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|.$$

\Rightarrow When two operators commute, there is no problem with simultaneous measurement. We will first prove Schwarz's inequality. Consider two states $|x\rangle$ and $|y\rangle$.

$$\langle x|x\rangle \langle y|y\rangle \geq \langle x|y\rangle \langle y|x\rangle$$

$$\Rightarrow \|x\| \|y\| \geq |\langle x|y\rangle|^2$$

To proceed, we define $|\alpha\rangle = |x\rangle + \lambda|y\rangle$, λ a constant to be determined. The length of this vector is

$$\langle \alpha|\alpha\rangle = \|x\| + \lambda\|y\|^2 \geq 0$$

$$\Rightarrow \langle \alpha|\alpha\rangle = \langle x|x\rangle + |\lambda|^2 \langle y|y\rangle + \lambda \langle x|y\rangle + \lambda^* \langle y|x\rangle \geq 0$$

This statement is true for all λ .

(4)

Set

$$\lambda = -\frac{\langle y|x\rangle}{\langle y|y\rangle} ; \quad \lambda^* = -\frac{\langle x|y\rangle}{\langle y|y\rangle}$$

$$\Rightarrow \langle x|x\rangle + \frac{\langle x|y\rangle\langle y|x\rangle}{\langle y|y\rangle} - \frac{\langle x|y\rangle\langle y|x\rangle}{\langle y|y\rangle} - \frac{\langle y|x\rangle\langle x|y\rangle}{\langle y|y\rangle} \geq 0$$

$$\Rightarrow \langle x|x\rangle\langle y|y\rangle \geq \langle y|x\rangle\langle x|y\rangle.$$

To go further, we define shifted operators

$$A_0 = \hat{A} - \langle \hat{A} \rangle \text{ and } B_0 = \hat{B} - \langle \hat{B} \rangle. \text{ Clearly } \langle A_0 \rangle = \langle B_0 \rangle = 0$$

$$\text{However, } \Delta A_0 = \Delta \hat{A} \text{ and } \Delta B_0 = \Delta \hat{B}. \text{ Also, } [A_0, B_0] = [\hat{A}, \hat{B}].$$

\Rightarrow we can work with the A_0 and B_0 operators of the \hat{A}, \hat{B} operators. Let $|x\rangle = A_0|\psi\rangle$ and $|y\rangle = B_0|\psi\rangle$.

We now apply Schwarz's inequality. We find that

$$\langle \psi | A_0^2 |\psi \rangle \langle \psi | B_0^2 |\psi \rangle \geq \langle \psi | A_0 B_0 |\psi \rangle \langle \psi | B_0 A_0 |\psi \rangle$$

$$\geq | \langle \psi | A_0 B_0 |\psi \rangle |^2.$$

For any complex number z , $|z|^2 \geq |\operatorname{Im} z|^2$

$$\operatorname{Im} z = \frac{1}{2i} [z - z^*].$$

$$| \langle \psi | A_0 B_0 |\psi \rangle |^2 \geq | \operatorname{Im} \langle \psi | A_0 B_0 |\psi \rangle |^2$$

$$\geq \frac{1}{4} | \langle [A_0, B_0] \rangle |^2$$

$$\langle A_0^2 \rangle \langle B_0^2 \rangle \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2$$

$$\Rightarrow \Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|.$$

for $\hat{A} = X, \hat{P} = \hat{B}, [X, P] = i\hbar/l$.

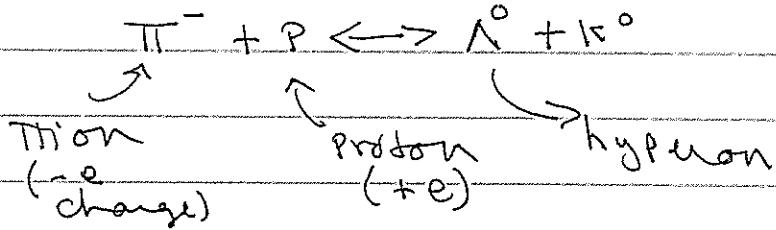
$$\Rightarrow \Delta X \Delta P \geq \hbar/2.$$

\Rightarrow uncertainty is tied to lack of commutativity.

We will show later why this could have been overshadowed.

3.) Neutral K-mesons

Neutral K-mesons (K^0) are produced in reactions of the form



The net charge on both sides vanishes.

K^0 and its antiparticle \bar{K}^0 have internal structure that we denote as hypercharge or strangeness, S .

For K^0 , $S = \pm 1$ while for \bar{K}^0 , $S = -1$.

In general strangeness is conserved in strong interactions but is broken in weak interactions ~ 10^{12} weaker than the ones we consider here.

(6)

Λ^0 has strangeness ≈ 0 . $S(\pi^+, p) = 0 \Rightarrow \Delta S = 0$ in this reaction. In high-energy collisions, we find a linear combination of K^0 and $\bar{K}^0 \Rightarrow S$ does not have a definite value in such a state.

We want to formulate a measurement made on such a superposition. Note first that $|K^0\rangle$ and $|\bar{K}^0\rangle$ form a complete basis for K-mesons.

$$S|K^0\rangle = |K^0\rangle, S|\bar{K}^0\rangle = -|\bar{K}^0\rangle$$

Let

$$|\Psi_1\rangle = \begin{pmatrix} |K^0\rangle \\ 0 \end{pmatrix}, |\Psi_2\rangle = \begin{pmatrix} 0 \\ |\bar{K}^0\rangle \end{pmatrix}.$$

$$\text{In this basis } S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of this matrix satisfy

$$(1-\lambda)(1+\lambda) = 0 \Rightarrow \lambda = 1, -1 \text{ as expected.}$$

Now we write S as

$$S = \sum_{i=1,2} |\Psi_i\rangle \langle \Psi_i| S \sum_{i=1,2} |\Psi_i\rangle \langle \Psi_i|$$

$$= |\Psi_1\rangle \langle \Psi_1| - |\Psi_2\rangle \langle \Psi_2|$$

$$= |\Psi_1\rangle \langle \Psi_1| - |K^0\rangle \langle \bar{K}^0|$$

In general we can write any operator

\hat{O} as

$$\hat{O} = \begin{pmatrix} \hat{\lambda} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \hat{\lambda} \end{pmatrix}$$

(7.)

$$\text{where } \hat{1} = \sum_i |i\rangle\langle i|$$

$$\Rightarrow \hat{0} = \sum_{i,l} |i\rangle\langle i| \hat{0} |l\rangle\langle l| \\ = \sum_i \alpha_i |i\rangle\langle i|$$

$$\text{where } \hat{0}|i\rangle = \alpha_i |i\rangle.$$

Another operator of interest is that for charge conjugation, C_P .

$$C_P |K^0\rangle = |\bar{K}^0\rangle.$$

$$C_P |\bar{K}^0\rangle = |K^0\rangle.$$

\Rightarrow in the K^0, \bar{K}^0 basis

$$C_P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\det(C_P - \lambda) = 0 \Rightarrow \lambda^2 - 1 = 0 \\ \Rightarrow \lambda = \pm 1.$$

The eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(8)

The eigenstates of CP are

$$|K_+\rangle = \frac{1}{\sqrt{2}} [|K^0\rangle + |\bar{K}^0\rangle]$$

$$|K_-\rangle = \frac{1}{\sqrt{2}} [|K^0\rangle - |\bar{K}^0\rangle].$$

$$CP |K_+\rangle = |K_+\rangle$$

$$CP |K_-\rangle = -|K_-\rangle.$$

Note $|K_{\pm}\rangle$ are not eigenstates of S.
 \Rightarrow CP eigenstates do not have definite strangeness. We can, of course, express the CP states in terms of the strange states.

$$|\psi_1\rangle = \begin{pmatrix} |K^0\rangle \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (|K_+\rangle + |K_-\rangle)$$

$$|\psi_2\rangle = \begin{pmatrix} 0 \\ |\bar{K}^0\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} (|K_+\rangle - |K_-\rangle).$$

next class, we will study the decay of K meson.