in the time domain is not always attractive. Electron repulsions and the phonon-mediated attraction act on different time scales and hence pair-binding of electrons is possible.

Let us evaluate the two-body scattering amplitude,

$$A_s = \langle \mathbf{p}_{4\sigma_1} \mathbf{p}_{3\sigma_2} | V_{ee} | \mathbf{p}_{1\sigma_1} \mathbf{p}_{2\sigma_2} \rangle, \tag{12.39}$$

where

$$V_{ee} = \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V(\mathbf{k}, \mathbf{k}' - \mathbf{q}) a^\dagger_{\mathbf{k}+\mathbf{q}} a^\dagger_{\mathbf{k}'-\mathbf{q}} a_{\mathbf{k}'} a_{\mathbf{k}}. \tag{12.40}$$

As discussed in Chapter 5, any two-body amplitude of this form separates into a difference of direct and exchange Coulomb integrals. For a general momentum-dependent potential, the direct and exchange terms will enter with different combinations of the momenta and spin. For example, the exchange term will be of the form $V(\mathbf{p}_1, \mathbf{p}_3)$ with $\sigma_1 = \sigma_2$, whereas the direct term will enter with no restriction on the spins and will depend on $V(\mathbf{p}_1, \mathbf{p}_4)$. Let us assume that the net attractive potential is a constant of the form

$$V_{ee} = \begin{cases} -V_o, & |\Delta\epsilon_{\mathbf{k}_F,\mathbf{q}}| < \hbar\omega_D, \\ 0, & |\Delta\epsilon_{\mathbf{k}_F,\mathbf{q}}| > \hbar\omega_D. \end{cases} \tag{12.41}$$

Because the potential is momentum-independent, the direct and exchange integrals are equal, and the scattering amplitude vanishes when $\sigma_1 = \sigma_2$. The only non-zero contribution arises from $\sigma_1 \neq \sigma_2$. As a consequence, for a constant interaction, the scattering amplitude is non-zero only if the electrons are locked into a singlet state. In this case, phonons induce a net attraction between electrons of opposite spin. The corresponding matrix element is of the form

$$\langle \mathbf{p}_{4\uparrow} \mathbf{p}_{3\downarrow} | V_{ee} | \mathbf{p}_{1\uparrow} \mathbf{p}_{2\downarrow} \rangle = -V_0 \delta_{\mathbf{p}_1+\mathbf{p}_2, \mathbf{p}_3+\mathbf{p}_4} \tag{12.42}$$

for scattering in the vicinity of the Fermi surface. Two particles locked into a singlet state will give rise to a gap at the Fermi level.

12.4 Cooper pairs

Consider now the somewhat artificial problem of a full Fermi sea containing N non-interacting electrons with two additional interacting electrons outside the sea. As a result of the Pauli exclusion principle, the momentum of the electrons outside the Fermi sea must exceed p_F. We take the potential of interaction to be the constant singlet pair potential derived in the previous section, Eq. (12.42). The spin wavefunction is hence antisymmetric with respect to spin. The corresponding spatial part must be symmetric to satisfy the overall antisymmetry requirement. The eigenvalue equation for our subsystem of two particles is

$$\left[-\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) + V(\mathbf{r}_1 - \mathbf{r}_2) - E \right] \Psi(\mathbf{r}_1, \mathbf{r}_2; \uparrow_1, \downarrow_2) = 0. \tag{12.43}$$

In general, the two-body wavefunction is a composite state

$$\Psi(\mathbf{r}_1, \mathbf{r}_2; \uparrow_1, \downarrow_2) = \psi(\mathbf{r}_1, \mathbf{r}_2) \chi_S(\uparrow_1, \downarrow_2), \tag{12.44}$$

where $\chi_S(\uparrow_1, \downarrow_2)$ represents the singlet spin state and ψ contains the spatial dependence. For two particles, we can express the spatial wavefunction in terms of the relative coordinate, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and a center of mass, $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, as

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \varphi(\mathbf{r}) e^{i\mathbf{Q}\cdot\mathbf{R}/\hbar}. \tag{12.45}$$

Likewise, the momenta of interest are the center of mass, $\mathbf{Q} = \mathbf{p}_1 + \mathbf{p}_2$, and the relative momentum, $\mathbf{q} = (\mathbf{p}_1 - \mathbf{p}_2)/2$. We expand $\varphi(\mathbf{r})$ in a Fourier series as

$$\varphi(\mathbf{r}) = \sum_{\mathbf{k}}{}' \frac{e^{i\mathbf{k}\cdot\mathbf{r}/\hbar}}{\sqrt{V}} \alpha_{\mathbf{k}}, \tag{12.46}$$

in which the prime indicates that the restriction $\mathbf{k} > \mathbf{k}_F$ is restricted to all states whose energy exceeds ϵ_F. Because $\mathbf{k}\cdot\mathbf{r} = \mathbf{k}\cdot\mathbf{r}_1 - \mathbf{k}\cdot\mathbf{r}_2$, we see that the pair state has momenta $(\mathbf{k}, -\mathbf{k})$. Note, if $\mathbf{k}_1 + \mathbf{k}_2 = 0$, then the center-of-mass motion drops out of the problem.

We focus first on the $\mathbf{Q} = 0$ solution. To this end, we define the Fourier components of the interaction potential,

$$V_{\mathbf{k}\mathbf{k}'} = \int \frac{d\mathbf{r}}{V} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}/\hbar} V(\mathbf{r}), \tag{12.47}$$

and introduce the center-of-mass Schrödinger equation

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{r})\right)\varphi(\mathbf{r}) = E\varphi(\mathbf{r}), \tag{12.48}$$

where μ is the reduced mass, $\mu = m/2$. Substituting the Fourier representation of $\varphi(\mathbf{r})$, multiplying by $\exp(-i\mathbf{k}\cdot\mathbf{r}/\hbar)$, and integrating, we obtain

$$(E - 2\epsilon_{\mathbf{k}})\,\alpha_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\alpha_{\mathbf{k}'} \tag{12.49}$$

as our new eigenvalue equation. Here, $\epsilon_{\mathbf{k}} = k^2/2m$. The matrix element $V_{\mathbf{k}\mathbf{k}'}$ is equivalent to

$$V_{\mathbf{k}\mathbf{k}'} = \langle \mathbf{k}, -\mathbf{k}\,|V_{ce}|\,\mathbf{k}', -\mathbf{k}'\rangle. \tag{12.50}$$

A typical scattering process in $V_{\mathbf{k}\mathbf{k}'}$ is shown in Fig. 12.10.

If we now introduce the approximation that

$$V_{\mathbf{k}\mathbf{k}'} = \begin{cases} -V_0, & k, k' > p_F, \\ 0, & \text{otherwise}, \end{cases} \tag{12.51}$$

the eigenvalue equation can be recast as

$$\begin{aligned} 1 &= -V_0 \sum_{k>k_F} \frac{1}{E - 2\epsilon_{\mathbf{k}}} \\ &= -V_0\phi(E). \end{aligned} \tag{12.52}$$

This equation is satisfied as long as $\phi(E) = -1/V_0$. The poles of $\phi(E)$ occur at $E = 2\epsilon_{\mathbf{k}}$, the total energy of the pair, which is bounded from below by $2\epsilon_F$. In a finite system, $\epsilon_{\mathbf{k}}$ takes on discrete values because $\mathbf{k}$ is quantized. As E approaches $2\epsilon_{\mathbf{k}}$ from below, $\phi(E)$ approaches $-\infty$. Just above $2\epsilon_{\mathbf{k}}$, $\phi(E)$ is $\sim +\infty$. For all $E < 2\epsilon_F$, $\phi(E)$ is negative. Hence,

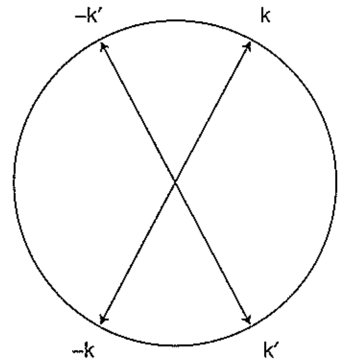

Fig. 12.10 Scattering between a pair of electron states across the Fermi surface.

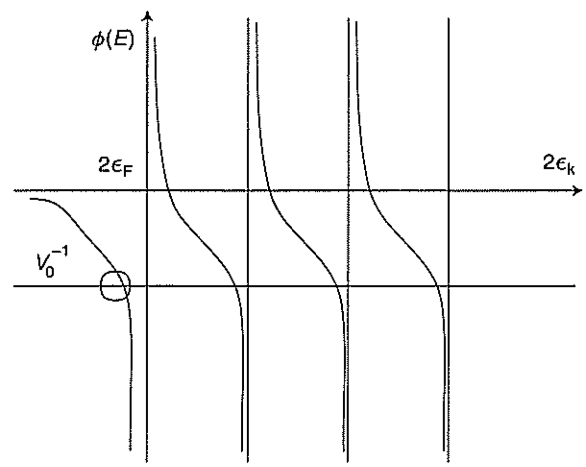

Fig. 12.11 A plot of $\phi(E)$ versus ϵ_k in the Cooper-pair problem. The intersection of $\phi(E)$ with the straight line $-V_0^{-1}$ determines the bound-state solutions for the pair.

a bound state forms when $\phi(E)$ crosses $-1/V_0$ for $E < 2\epsilon_F$. The intersection of $-1/V_0$ with $\phi(E)$ is illustrated graphically in Fig. 12.11. The existence of such a solution is the Cooper (C1956) pair problem. To find the precise energy of the bound state, we convert the sum in Eq. (12.52) to an integral,

$$1 = V_0 \sum_{\epsilon(\mathbf{k})}' \frac{1}{2\epsilon_\mathbf{k} - E} \rightarrow V_0 \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} \frac{N(x)\mathrm{d}x}{2x - E} \tag{12.53}$$

by introducing the density of states, $N(x)$. If we assume that $N(x)$ does not change significantly in the narrow range of integration, we can set $N(x) \approx N(\epsilon_F)$. Under these assumptions, the integral yields

$$-\frac{2}{V_0 N(\epsilon_F)} = \ln\left[\frac{\epsilon_F - \frac{E}{2}}{\epsilon_F - \frac{E}{2} + \hbar\omega_D}\right], \tag{12.54}$$

which implies that

$$\left(\epsilon_F - \frac{E}{2} + \hbar\omega_D\right)\exp\left(-\frac{2}{V_0 N(\epsilon_F)}\right) = \epsilon_F - \frac{E}{2}. \tag{12.55}$$

This linear equation can be solved immediately for the eigenenergy E:

$$E = 2\epsilon_F - \frac{2\hbar\omega_D \exp\left(-2/V_0 N(\epsilon_F)\right)}{1 - \exp\left(-2/V_0 N(\epsilon_F)\right)}. \tag{12.56}$$

In the limit that $2 \gg V_0 N(\epsilon_F)$, the exponential in the denominator can be expanded. The pair-binding energy

$$E \simeq 2\epsilon_F - 2\hbar\omega_D \exp\left(-\frac{2}{V_0 N(\epsilon_F)}\right) \tag{12.57}$$

in the weak-coupling limit results. Either of these expressions indicates that the Cooper pair is bound with an energy $< 2\epsilon_F$ whenever V_0 is non-zero and positive. This is a profound result. It implies that two electrons directly below the Fermi surface can lower their energy by being excited into a Cooper pair with momentum $(\mathbf{k}, -\mathbf{k})$ just above the Fermi surface provided that an attractive interaction of the form in Eq. (12.51) exists. This is known as the Cooper instability.

Further, we can estimate how the pair-binding energy depends on the center of mass of the Cooper pair. At the onset, we suspect that this quantity might scale as Q^2. We will show that this is not the case. To proceed, we extend the pair-binding criterion to the case in which $Q \neq 0$. For non-zero Q, the bare energy of the pair becomes $\epsilon_{\mathbf{q}+\mathbf{Q}/2} + \epsilon_{-\mathbf{q}+\mathbf{Q}/2}$. Consequently, the pair-binding condition becomes

$$1 = -V_0 \sum_{\mathbf{q}} \frac{1}{E - \epsilon_{\mathbf{q}+\mathbf{Q}/2} - \epsilon_{-\mathbf{q}+\mathbf{Q}/2}}. \tag{12.58}$$

For small Q, we can rewrite Eq. (12.58) as

$$1 = -V_0 \int_{\epsilon_F + v_F Q/2}^{\epsilon_F + v_F Q/2 + \hbar\omega_D} \frac{N(\epsilon_{\mathbf{q}})d\epsilon_{\mathbf{q}}}{E - 2\epsilon_{\mathbf{q}}}, \;, \tag{12.59}$$

dropping terms of $O(Q^2)$. The center-of-mass simply shifts the zero of the Fermi energy. The new pair-binding energy,

$$E = 2\epsilon_F + Q v_F - \frac{2\hbar\omega_D}{\exp\left(2/V_0 N(\epsilon_F)\right) - 1}, \tag{12.60}$$

is a linear function of the center-of-mass momentum. Translation of the center of mass strongly reduces the binding energy and could eventually break up the pair. To show this, we set $\epsilon_F = 0$ and evaluate the value of Q at which the Cooper pair loses most of its binding energy. We must then solve

$$\begin{aligned} Qv_F &= \frac{2\hbar\omega_D}{\exp\left(2/V_0 N(\epsilon_F)\right) - 1} \\ &\approx k_B T_c. \end{aligned} \tag{12.61}$$

Equivalently, $Q/\hbar \sim k_B T_c/\hbar v_F \sim 10^4\ \mathrm{cm}^{-1}$, which is roughly the reciprocal of the Pippard coherence length, $\xi \sim 10^{-4}$ cm. This is the effective radius of gyration of a Cooper pair, an enormous distance when compared to interatomic spacings. Such a large coherence length is a typical feature of phonon pairing mechanisms.

12.5 Fermi liquid theory

Of course, our problem is somewhat artificial in that we have ignored all interactions between the electrons save for the pair just above the Fermi surface. It is certainly reasonable to expect the simple picture of the Cooper instability to break down once we consider repulsive interactions among all of the electrons. That is, when electrons are interacting, we cannot *a priori* regard the non-interacting eigenstates as a valid description of our system (as we have done in the Cooper problem). We then are led to the question, is the instability real? We will answer this question by appealing to simple physical considerations arising from the scattering of electrons near the Fermi surface and a more formal argument involving the scaling of the full interacting Lagrangian. It is now well accepted that the normal state of a metal is described by Landau–Fermi liquid theory. In this account, it is claimed that the dominant effect of electron interactions in a metal is to renormalize the effective mass of the electron. The observed shift is on the order of 10 to 50 percent. Another essential claim of Fermi liquid theory is that there is a one-to-one correspondence between the excited states of the normal state of a metal and those of a non-interacting electron gas. The elementary excitations in Fermi liquid theory are called quasi-particles. A quasi-particle is a composite particle with a lifetime. The lifetime stems from collisions with other quasi-particles. When the lifetime (τ) of a quasi-particle is infinite, the state with such a particle is an eigenstate of the system. However, the minimum constraint that must hold for a quasi-particle state to be an eigenstate of a system is that $\hbar/\tau \ll \tilde{\epsilon}_p$, where $\tilde{\epsilon}_p$ is the energy of the quasi-particle. We will see below that as the energy of a quasi-particle approaches the Fermi level, its lifetime goes to infinity. The stability of quasi-particles at the Fermi level is a crucial tenent of Fermi liquid theory.

The vanishing of the scattering rate of electrons in the vicinity of the Fermi level can be shown as follows. Consider a near $T = 0$ distribution in which all but one of the electrons is below the Fermi surface. Let ϵ_1 be the energy of the electron above the Fermi surface. For an electron with this energy to scatter, it must interact with some electron with energy $\epsilon_2 < \epsilon_F$. The Pauli exclusion principle requires that after the scattering event, the electrons

To see this more clearly, we rewrite the relaxation rate as

$$T_1^{-1} = \frac{64\pi^3}{9\hbar}\mu_B'^2 |\alpha_N|^2 g^2(0) \int_\Delta^\infty d\varepsilon\, n(\varepsilon)\,(1-n(\varepsilon)) \left(\frac{\rho(\varepsilon)}{\varepsilon}\right)^2 \times \frac{(\varepsilon^2+\Delta^2)}{\left(1-H_{\mathrm{int}}^+ \mathrm{Re}G\right)^2 + \pi^2 |H_{\mathrm{int}}^+|^2 \rho^2(\varepsilon)}, \tag{12.318}$$

which reduces to

$$T_1^{-1} \propto \int_\Delta^\infty d\varepsilon\,(1-n(\varepsilon))\,n(\varepsilon)\,\frac{\varepsilon^2+\Delta^2}{\left(1+\pi^2 \left|H_{\mathrm{int}}^+\right|^2 g^2(0)\right)\varepsilon^2 - \Delta^2}. \tag{12.319}$$

In deriving Eq. (12.319), we ignored the real part of the Green function as it serves no relevant purpose as far as the convergence is concerned. This expression is completely convergent at $\varepsilon^2 = \Delta^2$. In fact, because $\left(1+\pi^2 |H_{\mathrm{int}}|^2 g^2(0)\right) > 1$, the integral does not diverge over the complete integration range. Consequently, summing high-order terms in the perturbation series results in a smearing of the peak in the relaxation rate immediately below T_c.

12.15 Josephson tunneling

Consider two superconductors separated from one another by a thin insulating barrier. Naively, we would expect no appreciable transport of charge between the two superconductors in the absence of an applied voltage, save possibly for single quasi-particle tunneling through the insulating barrier. Josephson (J1962) showed that this naive picture is not correct. In particular, he proved that in the absence of an applied voltage for a sufficiently thin barrier, Cooper pairs flow coherently between the two superconductors, thereby establishing a supercurrent through the barrier. Further, the transport of Cooper pairs across the barrier does not result in the creation of quasi-particles in either superconductor. When an applied voltage is present, the supercurrent oscillates with a well-defined period. The essence of both of these effects, dc and ac Josephson tunneling, rests in the phase coherence that obtains in the superconducting state.

We focus first on the dc Josephson effect. Consider two superconductors separated by a thin insulating barrier. Let H_T represent the Hamiltonian for single-particle tunneling across the thin barrier. The specific form of this term is not essential here. The only important feature is that H_T transfers only one electron at a time. We assume at the outset that there is no voltage difference between the two superconductors, and hence they are at the same chemical potential. We can derive the Josephson effect by making an analogy with electron transport in a 1d periodic chain in the tight-binding approximation. In this approximation, a single orbital is placed on each lattice site and a hopping term mediates transport among nearest-neighbor sites. In such a system, no energy is required to transport an electron across m lattice sites. Likewise, it requires no energy to translate m Cooper pairs across the barrier in the absence of an external voltage differential between the two superconductors. Let $|\Phi_{2m}\rangle$ represent the many-body state when $2m$ Cooper pairs are transferred across the barrier. The degeneracy of these states is split by the single-electron tunneling term, H_{T}. Consequently, we expand the total state of our system as a linear combination

$$|\Psi_\phi\rangle \equiv |\phi\rangle = \sum_m \mathrm{e}^{2\mathrm{i}m\phi}|\Phi_{2m}\rangle \tag{12.320}$$

over all such pair states. The phase ϕ plays the role of the wavevector k in the 1d periodic tight-binding model. As we have established earlier, the particle number, $2m$, and the phase, ϕ, are conjugate variables.

To compute the energy shift as a result of the tunneling processes, we employ perturbation theory. The first-order term, $\langle\phi|H_{\mathrm{T}}|\phi\rangle$, vanishes identically because ϕ is a sum of all pair states and H_{T} is a one-body operator. Consequently, the first non-zero term appears in second order. Let

$$\widehat{H}_{\mathrm{T}}^{(2)} = \widehat{H}_{\mathrm{T}}\frac{|I\rangle\langle I|}{E - E_{\mathrm{I}}}\widehat{H}_{\mathrm{T}} \tag{12.321}$$

represent the tunneling operator at second order with E_{I} the energy of the intermediate state, $|I\rangle$. The second-order correction to the energy,

$$\begin{aligned} E_\phi &= \langle\phi|\widehat{H}_{\mathrm{T}}^{(2)}|\phi\rangle \\ &= \sum_{m,m'} \mathrm{e}^{2\mathrm{i}\phi(m-m')}\langle 2m'|\widehat{H}_{\mathrm{T}}^{(2)}|2m\rangle \\ &= \sum_m \left(\mathrm{e}^{2\mathrm{i}\phi}\langle 2m|\widehat{H}_{\mathrm{T}}^{(2)}|2(m+1)\rangle + \mathrm{e}^{-2\mathrm{i}\phi}\langle 2m|\widehat{H}_{\mathrm{T}}^{(2)}|2(m-1)\rangle\right), \end{aligned} \tag{12.322}$$

is a sum of all matrix elements that differ by a single Cooper pair. We have assumed that $\langle\phi|\phi\rangle = 1$. To simplify this expression, we note that the energy of the intermediate state involves a particle–hole excitation, and hence E_{I} must exceed E by at least 2Δ. Consequently, $E - E_{\mathrm{I}} < 0$. If we regard the tunneling term to be purely real, we simplify the energy shift to

$$E_\phi = -\frac{\hbar J_0}{2}\cos 2\phi, \tag{12.323}$$

with

$$\hbar J_0 = 4\sum_m \left|\langle 2m|\widehat{H}_{\mathrm{T}}^{(2)}|2(m+1)\rangle\right|. \tag{12.324}$$

The minus sign in the energy shift arises from the sign of the excitation energy. From the Hamilton equation, Eq. (12.170), it is clear that if the energy shift depends on the phase, then the pair number fluctuates on either side of the barrier. This fluctuation is due entirely to the tunneling processes. We calculate the pair current directly,

$$I = 2e\frac{\mathrm{d}(2m)}{\mathrm{d}t} = 2e\left\langle\frac{\mathrm{d}E_\phi}{\mathrm{d}\hbar\phi}\right\rangle = 2eJ_0 \sin 2\phi, \tag{12.325}$$

by differentiating the energy shift with respect to the phase. Consequently, in the absence of an applied voltage, a dc supercurrent flows across the barrier. The value of the current ranges from $-2eJ_0$ to $2eJ_0$. A supercurrent of this form was first observed by Anderson and Rowell (AR1963). If a potential difference V exists across the barrier, then a term of the form $2mV$ must be added to the Hamiltonian. Consequently, from Hamilton's equations, Eq. (12.170), the phase fluctuates in time according to

$$\frac{\mathrm{d}(\hbar\phi)}{\mathrm{d}t} = 2eV. \tag{12.326}$$

Together, these two equations, Eq. (12.325) and Eq. (12.326), completely determine the behavior of the supercurrent across the barrier. To illustrate, consider the simplest case in which the voltage V is a constant in time. In this case, the phase ϕ varies linearly with time and, as a consequence, the current oscillates as $\sin(2eVt/\hbar)$. Hence, an alternating current flows with a frequency of $2eV/\hbar$.

Summary

We have shown that the pairing hypothesis of BCS is sufficient to account for all relevant experimental observables of low-temperature superconductors. In fact the BCS pairing mechanism is the only account available currently to describe the transition to a superconducting state. In contrast to low-T_c materials, superconductivity in the cuprates originates from doping an insulator. Further, the insulator possesses a partially-filled band and hence falls into the class of Mott insulators in which an absence of transport originates from strong electron repulsions. Consequently, we know *a priori* that we are not justified in starting from Fermi liquid theory to describe even the normal state properties. Simply stated, the deep phenomenology of the cuprates lies in the physics of doped Mott insulators. Whether a theory as succinct and crystal clear as the BCS account can be formulated for such systems remains to be seen.

Problems

12.1 Within the Ginsburg–Landau phenomenological approach, determine the form of the free energy density when a magnetic field is present. Show that the free energy difference between the superconducting and normal states is given by

$$F_S - F_N = -\frac{H_c^2(T)}{8\pi}. \tag{12.327}$$

12.2 Writing the Ginsburg–Landau wavefunction as $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$, show that the current density in terms of the variables θ and $n(\mathbf{r})$ is given by

$$\mathbf{j} = \frac{e\hbar}{m}\left(\nabla\theta - \frac{e\mathbf{A}}{c\hbar}\right)n(\mathbf{r}). \tag{12.328}$$

Now assume that in the bulk of a material, the current density vanishes. As a consequence, $\hbar\nabla\theta = e\mathbf{A}$. Integrate both sides of this expression around a closed loop in a superconducting ring and show that the resultant magnetic flux enclosed is quantized. What is the correct value of e for a superconductor?

12.3 Use second-order perturbation theory directly to show that the electron–phonon interaction is negative and given by the second term in Eq. (12.34).

12.4 Redo the Cooper pair instability calculation for triplet pairing between the electrons.

12.5 Evaluate $\langle r^2\rangle$ for a singlet Cooper pair.

12.6 In the problem of the instability of the superconducting state in the presence of the BCS pairing interaction, determine the form of the growth rate of the pair amplitude as $T \to T_c$.

12.7 Evaluate the commutator $[b_k, b_k^\dagger]$, where the $b_k's$ are the Cooper pair annihilation operators. What does the lack of commutativity of the Cooper pair creation and annihilation operators mean?

12.8 Calculate the average number of particles in a superconductor. Let $|\Psi\rangle$ represent the BCS pair state. Show that the average value of the number operator, N, is given by

$$\langle\Psi|N|\Psi\rangle = \langle\Psi|\sum_{k,\sigma} a_{k\sigma}^\dagger a_{k\sigma}|\Psi\rangle = 2\sum_k |v_k|^2 \tag{12.329}$$

in the pair state. Also evaluate the fluctuation $\langle(N - \langle N\rangle)^2\rangle$. You should obtain a simple result involving u_k and v_k only. For what special value of u_k and v_k is the fluctuation maximized? Interpret your result.

12.9 So far we have ignored any spatial inhomogeneities in the gap. Consider a gap of the form $\Delta_\mathbf{q} = \Delta_0 e^{2i\mathbf{q}\cdot\mathbf{r}}$, where $q \ll p_F$. Find the new self-consistent condition for Δ_0. At $T = 0$, show that Δ is independent of q for $q < q_c \approx \Delta_0/\hbar v_F$. Near T_c, expand the gap equation to find that

$$\frac{\Delta(T)}{k_B T_c} \approx \frac{8\pi^2}{7\zeta(3)}(1 - T/T_c) - \frac{2}{3}\left(\frac{\hbar^2 p_F}{m k_B T_c}\right)^2 q^2. \tag{12.330}$$

Then determine the critical value of q that makes the gap vanish.

12.10 Evaluate the sums explicitly in Eq. (12.260) and show that for $T/T_c \ll 1, F_N - F_S \propto 1 - (T/T_c)^2$.

12.11 An Anderson-type impurity is placed in a superconductor. You are to formulate this problem and develop a criterion for local moment formation. There are a number of assumptions that can be applied. First, when you transform to the quasi-particle basis, ignore all terms that do not conserve spin and particle number. The problem should now be straightforward. You should be able to redo the Anderson problem completely. Discuss clearly when the local moment exists and when it does not.

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