There are only two distinct solutions to this equation, $\theta = 0$ and $\theta = \pi$. The former corresponds to the standard band insulator and the latter to the topological insulator. We can express this simply as

$$\theta = \pi \; (m \bmod 2) \tag{15.81}$$

and hence θ is the key Z_2 invariant that allows us to distinguish traditional insulators from band insulators. What does this mean physically? Applying a field produces a flux which in turn gives rise to a Hall conductance. The normalization we have chosen such that θ is the Z_2 invariant tells us immediately that the Hall conductance for a single Dirac cone is

$$\sigma_{xy} = \frac{1}{2}\frac{e^2}{h}. \tag{15.82}$$

In effect, $\theta = \pi$ implies that there is only a single Dirac cone on the surface of a topological insulator and hence such states are distinct from a single layer of carbon, graphene, which has an even number. Although this formulation has a drawback in that it assumes a gap exists, it does serve to illustrate clearly that the Z_2 invariance of the magneto-electric term in the action underlies the physics of topological insulators.

15.6 Laughlin liquid

In the original experiments of Tsui, Störmer, and Gossard (TSG1982), the Hall conductance exhibited a sharp plateau at the value of $e^2/3h$. In later experiments, fractional quantization of the Hall conductance was also observed for values of $4/3$, $5/3$, $7/3$, $1/5$, $2/5$, $3/5$, $7/5$, $8/5$, etc. The common ingredient in these sequences is the presence of the odd denominator. The original challenge in explaining these experiments, however, lay not so much in accounting for the fractional value of the Hall conductance but rather in explaining the nature of the electronic state that exhibited the fractional quantization. For example, a non-interacting model in which the lowest Landau level is fractionally occupied at a filling of ν will exhibit a conductance $\nu e^2/h$. However, when a Landau level is fractionally occupied, electrons can scatter into the empty states and hence longitudinal transport will not be dissipationless; as a consequence, $\sigma_{xx} \neq 0$. In addition, the persistence of a plateau at fractional filling indicates that somehow an appropriately partially occupied Landau level is stable to even the lowest-lying excitations. That is, an energy gap separates the ground state from all excited states. In the absence of electron interactions, the energy cost to add an additional electron to a partially filled Landau level is essentially zero. This would suggest that as the field is changed, sharp plateaus in the conductance should desist in the non-interacting model for a partially filled Landau level.

Consequently, the fractional quantum Hall state cannot be understood without including the role of electron interactions. However, at the outset, it is not clear what this state should look like. Thus far, we have introduced two electronic states that arise fundamentally from electron–electron interactions: the Wigner crystal and the superconducting state. While the vanishing of the longitudinal resistance suggests that the fractional quantum Hall state bears some resemblance to a superconducting state, it is not clear how such a state would survive a large perpendicular magnetic field. What about the Wigner crystal? As mentioned in Chapter 5, because a magnetic field freezes the electron zero-point motion, Wigner crystallization is stabilized. In fact, in the limit in which the interparticle separation is large relative to the cyclotron radius (magnetic length), electron correlations dominate and the conditions for Wigner crystalization become favorable. However, a Wigner crystal does not exhibit dissipationless transport as a threshold voltage must be applied before transport obtains. We suspect then that the resolution of the fractional quantum Hall state lies elsewhere.

Indeed it does. After Laughlin (L1983; L1987), we consider an interacting electron gas in the presence of a perpendicular magnetic field

$$H = \sum_i \left[\frac{1}{2m} \left(\mathbf{p}_i - \frac{e}{c}\mathbf{A}_i \right)^2 + V(\mathbf{r}_i) \right] + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{15.83}$$

where $V(\mathbf{r})$ is the compensating neutralizing potential from the ions and the sums over i and j are over the electrons. Fractional quantization of the Hall conductance is observed at a field of 15 T. At this field, the cyclotron energy is roughly three times the Coulomb energy, e^2/ℓ. Hence, it should be a fairly good approximation to use the non-interacting eigenstates of the lowest Landau level as a starting basis for constructing the true many-body wavefunction. In general, the non-interacting eigenstates are products of a polynomial in the electron coordinate and a Gaussian. For an interacting system, we expect that the true many-body state will involve differences of the electron coordinates. Let us define the complex electron coordinate, $z = x + iy$. We consider the ansatz,

$$\Psi_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{1 \le j < k \le N} f(z_j - z_k) \exp\left(-\sum_{j=1}^{N} \frac{|z_j|^2}{4\ell^2} \right), \tag{15.84}$$

where $f(z_j - z_k)$ is a polynomial in the electron coordinates. After studying the two-electron problem described by Eq. (15.83), Laughlin found that $f(z_1 - z_2) = (z_1 - z_2)^{2p+1}$, where p is an integer. Because the exponent $2p + 1$ is odd, the wavefunction is antisymmetric with respect to interchange of two electrons. Analogous results were also obtained for the equivalent three-electron problem (L1983). Hence, Laughlin proposed that

$$\Psi_m(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{1 \le j < k \le N} (z_j - z_k)^m \exp\left(-\sum_{j=1}^{N} \frac{|z_j|^2}{4\ell^2} \right) \tag{15.85}$$

must accurately describe the ground state of a fractional quantum Hall system. While this wavefunction was originally argued to be a variational state (where m must be determined), the overlap of this wavefunction with the exact eigenstate for small clusters of electrons is typically greater than 99 percent for interaction potentials of the form $u(r) = 1/r, -\ln r$, and $\exp(-r^2/2)$. This would suggest that the variational character of Ψ_m is minimal, and m must be determined from a fundamental principle. It turns out that Ψ_m is an eigenstate of the total angular momentum operator with eigenvalue

$$M = \frac{N(N-1)m}{2}. \tag{15.86}$$

That M is the total angular momentum of the operator L_z follows from expanding the product over $z_j - z_k$ in Eq. (15.84) and realizing that there are at most $N(N-1)m/2$ factors of z_i in each term. If the differential form of the L_z operator is applied to such a product, the eigenvalue M results. As a consequence, we can think of the Laughlin state as being a superposition of states within the lowest Landau level with the same angular momentum. This removes completely the variational character of the Laughlin state.

To understand precisely what physics the Laughlin state describes, we write the square of this state

$$|\Psi_m|^2 = \exp[-\beta\Phi] \tag{15.87}$$

in terms of a Boltzmann weight, where Φ is the interaction energy,

$$\Phi(z_1, \ldots, z_N) = -2 \sum_{1 \le j < k \le N} \ln|z_j - z_k| + \frac{1}{2m\ell^2} \sum_{j=1}^{N} |z_j|^2 \tag{15.88}$$

and $\beta = m$ plays the role of the inverse temperature. Equation (15.88) is exactly the interaction energy for a one-component plasma (such as an electron gas) consisting of N identical charges with charge $\sqrt{2}$. The second term in Eq. (15.88) represents the interaction energy with the neutralizing background, $U_b(z_i) = |z_i|^2/(2m\ell^2)$. To see this clearly, we note that the potential for the compensating background should satisfy a Poisson equation, $\nabla^2 U_b(z_i) = 4\pi\rho$, where ρ is given by $\nu n_B = \nu/(2\pi\ell^2)$ (see Eq. (15.21)). Performing the differentiation in the Poisson equation reveals that the compensating charge density per unit area,

$$\rho = \frac{1}{2\pi\ell^2 m} \equiv \frac{\nu}{2\pi\ell^2}, \tag{15.89}$$

is exactly the uniform electron charge density in the lowest Landau level if we identify the Landau filling factor ν with $1/m$. The Laughlin state accurately describes the ground state of an N-electron system for density and magnetic field strengths such that the filling in the lowest Landau level is given by $\nu = 1/m$. As a system with a uniform electron density, the Laughlin state is distinct from an electron crystal state, such as a Wigner crystal. In fact, electrons condensing into the Laughlin state do so without breaking any symmetries. Hence, a Landau-type description presented in the previous chapter, which necessarily involves the symmetry breaking of some order parameter, is not possible for the formation of fractional quantum Hall states. The importance of the Laughlin state in the development of the fractional quantum Hall effect cannot be overestimated.

We are now poised to explore the excitations of the Laughlin liquid. For filling fractions ν different from $1/m$, excitations emerge. Consider changing the filling by piercing the sample at z_0 with an infinitely thin magnetic solenoid. Although ν is now slightly less than $1/m$, the electrons will attempt to stay in the state Ψ_m. However, they cannot do this without diminishing the charge density at the insertion point of the magnetic solenoid. We can simulate such a depletion by excluding the electrons from z_0. Consequently, we anticipate that

$$\begin{aligned}\Psi_m^+(z_0; z_1, \ldots, z_N) &= \prod_{j=1}^{N}(z_j - z_0)\Psi_m(z_1, \ldots, z_N)\\ &= A_{z_0}^+\Psi_m(z_1, \ldots, z_N)\end{aligned} \tag{15.90}$$

might describe the wavefunction for the new many-body state with a "quasi-hole" at z_0. Indeed it does, as shown by Laughlin (L1983). A more quantitative argument for the quasi-particle wavefunction in Eq. (15.90) stems from noting that if the solenoid carries flux ϕ, each single-particle state is changed accordingly,

$$z^n e^{-\frac{|z|^2}{4\ell^2}} \to z^{n+\alpha} e^{-\frac{|z|^2}{4\ell^2}}, \tag{15.91}$$

to accommodate the additional flux. Here $\alpha = \phi/\phi_0$, with $\phi_0 = hc/e$ the flux quantum. If the solenoid carries one quantum of flux, then the prefactor of the Gaussian is z^{n+1}. That each single-particle state is now multiplied by an extra factor of z supports the ansatz for the quasi-particle wavefunction, Eq. (15.90). To utilize the plasma analogy, we square the quasi-particle wavefunction to find that, aside from a background normalization factor of $|z_0|^2$, the energy of the many-body state with a "quasi-hole",

$$\Phi_{qp}(z_0; z_1, \ldots, z_N) = \Phi(z_1, \ldots, z_N) - \frac{2}{m}\sum_{j=1}^{N}\ln|z_j - z_0|, \tag{15.92}$$

is that of a one-component plasma interacting with a charge fixed at z_0. The magnitude of the charge is $1/m$ or, in electron units, e/m. Likewise, if the solenoid were to extract a single flux quantum, a "quasi-electron" would be created with charge $-e/m$. Numerically, the energy to create or destroy quasi-particles in a three-electron fractional quantum Hall state is roughly 4 K at a field of 15 T (L1983). Improved estimates of the gap in the fractional quantum Hall state were obtained by Girvin, MacDonald, and Platzman (GMP1985). In fact, their work was pivotal in establishing that all collective excitations from the Laughlin state have a finite energy gap. Hence, the Laughlin state does satisfy the criterion of having a gap to all excitations. The excitation energy can be thought of as the Coulomb energy required to place a particle of charge $\pm e/m$ in the quantum liquid. The distance over which the charge acts is proportional to the magnetic length, $\propto \ell$. Hence, the excitation energy should scale as $e^2/\ell \propto \sqrt{B}$ and thus vanishes in the absence of an applied magnetic field. It is the presence of the energy gap in the excitation spectrum that makes the Laughlin state incompressible and ultimately leads to dissipationless transport.

Nonetheless, some experiments have revealed the existence of gapless excitations (ASH1983; MDF1985) in 2d quantum Hall systems. These excitations are believed to live on the edge of quantum Hall systems as they are explicitly excluded from the bulk by Laughlin's gauge argument. For integer quantum Hall states, the edge excitations are well described by Fermi liquid theory as electron interactions are relatively unimportant in the integer effect. However, in the fractional quantum Hall effect, the situation is entirely different, as we have seen. Quantum mechanical states confined to move at the edge of a fractional quantum Hall system are essentially 1d strongly correlated electron systems. We have shown in Chapter 10 that electron interactions in a 1d system give rise to Luttinger rather than Fermi liquid behavior. As a result of the chirality of the edge current, edge states in the fractional quantum Hall effect are chiral Luttinger liquids (W1990). They exhibit all the properties indicative of Luttinger liquids discussed in Chapter 9, including an excitation spectrum that vanishes algebraically in the vicinity of the Fermi energy. Figure 10.6 confirms the algebraic dependence of the excitation spectrum in the edge of the $\nu = 1/3$ quantum Hall state, thereby putting the chiral Luttinger liquid model for the edge states in the fractional quantum Hall effect on firm experimental footing.

Consider for the moment the problem of the statistics associated with interchanging (A1985; F1992) two quasi-holes or two quasi-electrons in a fractional quantum Hall state. The wavefunction describing such pair excitations, which we will locate at $z = z'$ and $z = z''$, is analogous to Eq. (15.90) except the product $A^{\pm}_{z'}A^{\pm}_{z''}$ now multiplies the Laughlin state Ψ_m. The problem we address is, under interchange of two quasi-particles such that

$$A^{\pm}_{z'}A^{\pm}_{z''}\Psi_m(z_1, \ldots, z_N) = e^{i\phi}A^{\pm}_{z''}A^{\pm}_{z'}\Psi_m(z_1, \ldots, z_N), \tag{15.93}$$

what phase, ϕ, does the wavefunction incur? For interchange of electrons, $\phi = \pi$ and for bosons, $\phi = 2\pi$ or, equivalently, 0. We will show now that the phase change for the interchange of two quasi-particles in the Laughlin state is $1/m$ and hence fractional.

Quite generally, the wavefunction of a particle traversing a closed loop in the presence of a vector potential will acquire the phase

$$i\gamma = i\frac{q}{\hbar c}\oint d\ell \cdot \mathbf{A}. \tag{15.94}$$

We will take the vector potential to be the generator of the magnetic field B felt by the particle. Consequently, the line integral that determines the phase is simply equal to the field times the surface area enclosed by the path. Let R be the radius of the loop. As a result, the phase change is $\gamma = q\pi R^2 B/\hbar c$. Recalling that the magnetic field is related to the electron density through $\rho = \nu eB/\hbar c$, and the quasi-particle charge is $e\nu$, we find that the total phase encountered is $\gamma = 2\pi N$, where N is the number of electrons in the system. This is a key result as the phase is proportional to the number of electrons enclosed in the loop. Now let us redo the argument assuming that amidst the sea of electrons lies a quasi-hole of charge