Superconducting magnetic field sensors: SQUID

SQUID – superconducting quantum interference device

SQUID helmet project at Los Alamos

https://www.newswise.com/articles/los-alamos-unveils-new-brain-imaging-system

Magnetic field scales:

Earth field: ~1G

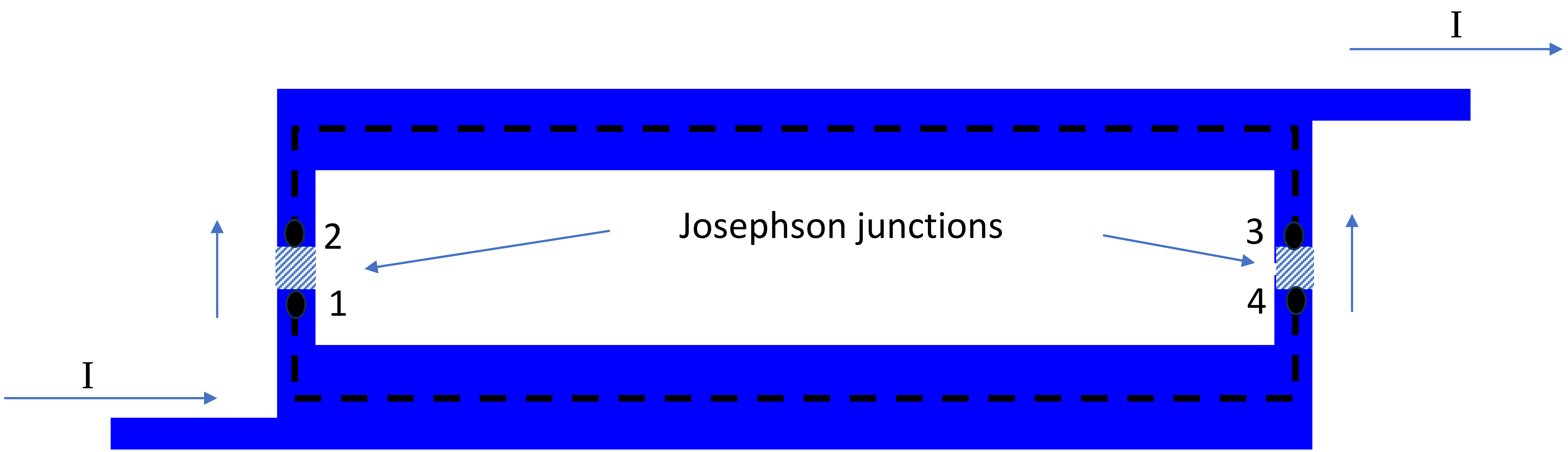
Fields inside animals: ~0.01G-0.00001G

Fields of the **human brain**: ~0.3nG
This is less than a hundred-millionth of the Earth's magnetic field.

NanoGallery.info - Superconducting Helmet. SQUID used to measure brain magnetic fields

SQUIDs, or Superconducting Quantum Interference Devices, invented in 1964 by Robert Jaklevic, John Lambe, Arnold Silver, and James Mercereau of Ford Scientific Laboratories, are used to measure extremely small magnetic fields. They are currently the most sensitive magnetometers known, with the noise level as low as 3 $fT \cdot Hz^{-1/2}$. While, for example, the Earth magnet field is only about 0.0001 Tesla, some electrical processes in animals produce very small magnetic fields, typically between 0.000001 Tesla and 0.000000001 Tesla. SQUIDs are especially well suited for studying magnetic fields this small.
Measuring the brain's magnetic fields is even much more difficult because just above the skull the strength of the magnetic field is only about 0.3 picoTesla (0.0000000000003 Tesla). This is less than a hundred-millionth of Earth's magnetic field. In fact, brain fields can be measured only with the most sensitive magnetic-field sensor, i.e. with the superconducting quantum interference device, or SQUID.

Superconducting quantum interference device: SQUID



Here the blue color is a bulk superconductor, the white color represents empty space, dashed region is the insulator through which electrons transit by quantum tunnel.
The dashed black line is a contour over which we calculate the phase change. The phase accumulation on a closed contour like this one must be zero or 2πn, where n is an integer. This is required because the wavefunction must be a single-value function.

Since the wave function is a single-valued function, the total phase accumulation must satisfy the following formula:

$$\Delta\varphi_{21} + \Delta\varphi_{32} + \Delta\varphi_{43} + \Delta\varphi_{14} = 2\pi n$$

Superconducting condensate described as a quantum system

Superconducting Condensate:

Reminder: The time-dependent Schrodinger equation is: $i\hbar \frac{d}{dt}\Psi = \hat{H}\Psi = E\Psi$

If E=0 then the wave function is constant. This is the key for understanding superconductivity.

This equation is originally developed for a single electron. In a superconductor, the number of superconducting electrons is very large and they all act in unison, i.e. coherently. One way to understand superconductivity is to assume that all the electrons glue together into a superconducting condensate which then act as just one quantum particle described by a wave function Ψ. Although in the quantum devices, such as qubits, superpositions of states with different energies is possible, in most superconducting devices, such as solenoids and SQUIDs the energy of the system is well defined, i.e. the equation $\hat{H}\Psi = E\Psi$ is valid. So, in this lecture we assume that superconducting electrons are in their ground state, which is a stationary state.

-If no voltage is applied, The energy E is usually the minimum possible energy (ground state). Thus, finding the wavefunction of superconducting electrons is equivalent to finding the ground state of one electron.

The ensemble of all superconducting electrons, i.e., those which have the same wavefunction, is called "superconducting condensate". It can be treated as just one electron, except that the normalization is different: The electronic density equals the square of the wave function

$$n_s = |\psi|2$$

Thus, to understand superconductivity at the very basic level we can think of just one electron, solve the corresponding time-independent Schrodinger equation (with the vector potential included), find the wave function and normalize the wave function to the density of the superconducting electrons, which is material-specific. Using the wave function we can calculate all other physical quantities of interest, such as the electric current for example.

Supercurrent, superfluid velocity, charge density

Superconducting Condensate: Since there are many superconducting electrons in a typical superconductor device, and all of these electrons are described by the same wave function (all are "coherent"), therefore the module squared of the wave function equals the density of the superconducting electrons $n_s(\boldsymbol{r}) = |\psi(\boldsymbol{r})|^2$. Such density is sometimes called "superfluid density". Thus, the density of the electric charge is $\rho(\boldsymbol{r}) = e^*|\psi(\boldsymbol{r})|^2$, where e^* is the effective charge of superconducting electrons, which is twice the charge of one electron (-*e*), i.e, $-e^* = -2e$ or $e^* = 2|e| > 0$. The most important parameter is the electric current. The electric current density, **j**(**r**), is the charge density multiplied by the velocity of the superconducting condensate, i.e., **j**(**r**)=**v**(**r**)$\rho(\boldsymbol{r})$= **v**(**r**) e^* $|\psi(\boldsymbol{r})|^2$. To define the electric current density, **j**, we need to know the superfluid velocity, **v**, and the superfluid density, n_s. These quantities can be calculated if the wavefunction in know.

The simplest wave function is the plane-wave wave-function. It describes homogeneous supercurrent in a superconductor. Mathematically it is written as: ψ= $\psi_0$exp[i**kr**]=$\psi_0$exp[i**pr**/ $\hbar$]. If there is no voltage, then E=0, then the quantum state is not changing in time. Therefore, we use a time independent wave function in many cases.

In classical electrodynamics it is known that the canonical momentum of a charge particles is determined as:

$$\boldsymbol{p} = m^*\boldsymbol{v} - e^*\boldsymbol{A}$$

Diamagnetic graphite levitation

And the Hamiltonian (explained in next two slides) is:

$$H = \frac{m^*\boldsymbol{v}^2}{2} = \frac{(m^*\boldsymbol{v})^2}{2m^*} = \frac{(\boldsymbol{p} + e^*\boldsymbol{A})^2}{2m^*}$$

Here the charge of the superconducting particle is take as –e*

https://www.imagesco.com/magnetism/graphite-levitation-kit.html

Charged particle in magnetic field: classical physics

2 Lorentz Force Law

The Lorentz force in Gaussian Units is given by:

$$\vec{F} = Q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right), \tag{4}$$

where Q is the electric charge, $\vec{E}(\vec{x},t)$ is the electric field and $\vec{B}(\vec{x},t)$ is the magnetic field. If the sources (charges or currents) are far away, $\vec{E}$ and $\vec{B}$ solve the homogeneous Maxwell equations. In Gaussian Units, they are given by

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{6}$$

The magnetic field $\vec{B}$ can be derived from a *vector potential* $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{7}$$

$$\vec{F} = Q\left(-\vec{\nabla}\varphi - \frac{1}{c}\frac{d\vec{A}}{dt} + \frac{1}{c}\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right)\right)$$

So, the Lagrangian for a particle in an electromagnetic field is given by

$$L = \frac{1}{2}mv^2 - Q\,\varphi + \frac{Q}{c}\,\vec{v}\cdot\vec{A} \tag{26}$$

4.1 The Hamiltonian for the EM-Field

We know the canonical momentum from classical mechanics:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{27}$$

Using the Lagrangian from Eq. (26), we get

$$p_i = mv_i + \frac{Q}{c}\,A_i \tag{28}$$

The Hamiltonian is then given by

$$H = \sum_i p_i \dot{x}_i - L = \frac{1}{2}mv^2 + Q\,\varphi, \tag{29}$$

where v resp. $\dot{x}$ must be replaced by p: Solving Eq. (28) for v_i and plugging into Eq. (29) gives

$$H = \frac{1}{2m}\left|\vec{p} - \frac{Q}{c}\,\vec{A}\right|^2 + Q\,\varphi \tag{30}$$

So the kinetic momentum in is in this case given by

$$\vec{P} = m\vec{v} = \vec{p} - \frac{Q}{c}\,\vec{A} \tag{31}$$

notes09c.pdf (sunysb.edu)

http://insti.physics.sunysb.edu/itp/lectures/01-Fall/PHY505/09c/notes09c.pdf

Charged particle in magnetic field: classical physics

1 Introduction

Conservative forces can be derived from a Potential $V(q,t)$. Then, as we know from classical mechanics, we can write the Lagrangian as

$$L(q,\dot{q},t) = T - V, \tag{1}$$

where T is the kinetic energy of the system. The Euler-Lagrangian **equations of motion** are then given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2}$$

In three dimensions with cartesian Coordinates, this can be written as

$$\frac{d}{dt}\left(\vec{\nabla}_{\vec{v}} L\right) - \vec{\nabla} L = 0. \tag{3}$$

Here, $\vec{\nabla}_{\vec{v}}$ means the gradient with respect to the velocity coordinates.

Now we generalize $V(q,t)$ to $U(q,\dot{q},t)$ – this is possible as long as $L = T - U$ gives the correct equations of motion.

If we identify $m\frac{d\vec{v}}{dt}$ with the force $\vec{F}$, given by Newton's Law, we can solve Eq. (20) for $\vec{F}$:

$$\vec{F} = Q\left(-\vec{\nabla}\varphi - \frac{1}{c}\frac{d\vec{A}}{dt} + \frac{1}{c}\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right)\right) \tag{21}$$

which is just the correct expression for the Lorentz Force Law, given by Eq. (16).

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Transition to quantum mechanics: Mathematic description of the supercurrent

In QM we replace momentum with momentum operator.
Velocity operator in quantum mechanics is: $m^*\hat{v} = -i\hbar\boldsymbol{\nabla} + e^*\boldsymbol{A}$

Here m^* is the effective mass of the charge carriers and e^* is the effective charge carriers in the superconductor.
Since electrons form pairs in the superconductor, we may choose
m^*=2m and e^*=2e.

Magnetic field and vector-potential: $\boldsymbol{B} = curl\boldsymbol{A}$

Consider the case when the wavefunction such that its amplitude is approximately constant: $|\psi| = \psi_0$

Therefore, the wave function can be written as: $\psi = \psi_0 e^{i\varphi} = \psi_0 e^{ipx/\hbar}$ The phase, φ, is a function of coordinates.

Expression for the **supercurrent density**: $\boldsymbol{j} = e^* n_s \boldsymbol{v} = (e^*/m^*)\, n_s m^* \boldsymbol{v}$

Now let us calculate the **superfluid velocity**, $\boldsymbol{v}$, mean value (“expectation value”):

$$n_s m^* \boldsymbol{v} = \psi^*(m^*\hat{v})\psi = \psi_0 e^{-i\varphi}(-i\hbar\boldsymbol{\nabla} + e^*\boldsymbol{A})\psi_0 e^{i\varphi}$$

Note that the superconducting electron density is: $n_s = \psi^*\psi = \psi_0^2$

$$m^*\boldsymbol{v} = e^{-i\varphi}(\hbar\boldsymbol{\nabla}\varphi + e^*\boldsymbol{A})e^{i\varphi} = \hbar\boldsymbol{\nabla}\varphi + e^*\boldsymbol{A}$$

Correspondingly, the phase gradient is: $\boldsymbol{\nabla}\varphi + \frac{e^*\boldsymbol{A}}{\hbar} = \frac{m^*}{\hbar e^* n_s}\boldsymbol{j} = \frac{m^*}{\hbar}\boldsymbol{v}$

Equation for the supercurrent: $\boldsymbol{j} = e^* n_s \boldsymbol{v} = (e^*/m^*) n_s(\hbar\boldsymbol{\nabla}\varphi + e^*\boldsymbol{A})$

Gauge invariance

Gauge transformation: Suppose we change the vector potential by a gradient of a function, as follows:

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi$$

With such transformation the vector potential the magnetic field remains the same: $B' = curl(\boldsymbol{A} + \boldsymbol{\nabla}\chi) = curl\boldsymbol{A} = \boldsymbol{B}$

With such gauge transformation all physical quantities remain unchanged, including the velocity. To calculate the velocity at each point in the superconductor we need to use the collective wave function of superconducting electrons. With the gauge transformation presented above the phase of the wavefunction also changes. The change of the phase must compensate the transformation of the vector potential in such a way that the local superfluid velocity is not changed.

Under the gauge transformation, the phase of the wave function is changed to φ'. From the previous slide,

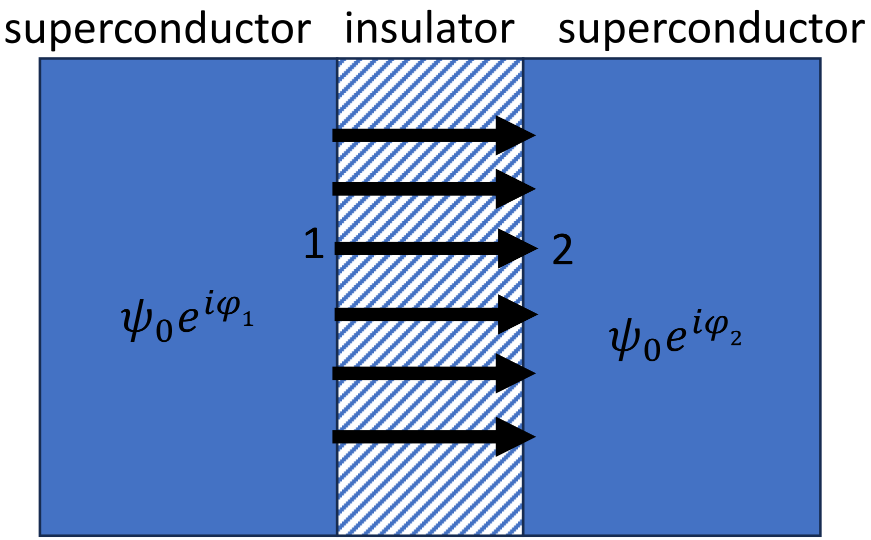
$$\frac{m^*}{\hbar e^* n_s}\boldsymbol{j} = \frac{m^*}{\hbar}\boldsymbol{v} = \boldsymbol{\nabla}\varphi + \frac{e^*\mathbf{A}}{\hbar} = \boldsymbol{\nabla}\varphi' + \frac{e^*\mathbf{A}'}{\hbar} \Rightarrow \boldsymbol{\nabla}(\varphi' - \varphi) + \frac{e^*}{\hbar}(\mathbf{A}' - \mathbf{A}) = 0,$$

which is simplified to $\boldsymbol{\nabla}\left(\varphi' - \varphi + \frac{e^*\chi}{\hbar}\right) = 0$, and so

$$\varphi' = \varphi - \frac{e^*}{\hbar}\chi$$

Josephson junction (JJ) in zero magnetic field

JJ is formed by two superconductor placed very close to each other but separated by a very thin (a few Angstroms) insulating barrier.



Supercurrent is defined by the phase difference $\Delta\varphi = \varphi_2 - \varphi_1$ between the two superconductors:

$$I_s = I_c \sin(\Delta\varphi) = I_c \sin(\Delta\varphi_{21})$$

This current is flowing through the insulator barrier without any voltage applied. Here I_c is the critical current, which is a constant for a given junction.

A supercurrent I can flow, horizontally, from the left superconductor to the superconducting block on the right through the insulating barrier (dashed region). The current is due to the quantum tunneling effect. The formula for the current (in the case when vector potential is zero) is: $I = I_c \sin \Delta\varphi$ where $\Delta\varphi$ is the phase difference and I_c is the critical current, i.e., the maximum possible supercurrent. Note that the current is controlled by the phase difference and not by the voltage as in normal circuits.

The arrow shows the trajectory over which we calculate the phase difference, as follows:

$$\Delta\varphi = \int_1^2 \boldsymbol{\nabla}\varphi d\boldsymbol{l} = \varphi_2 - \varphi_1$$

The formula above is true when the magnetic field is zero and the vector potential is zero. Also, it assumes the phase is constant within the bulk of the superconductor.

Josephson junction (JJ) in presence of magnetic field

The phase difference,Δφ, depends on the choice of the gauge. Yet, the supercurrent must not depend on the choice of the gauge. Therefore, the expression for the supercurrent needs to be modified as follows:

$$I = I_c \sin \Delta\gamma$$

Where $\Delta\gamma$ is so-called gauge invariant phase difference. In the example we consider in the previous slide $\Delta\gamma = \Delta\varphi$ in the gauge choice in which vector potential is zero. Now we transition to a different gauge, $\boldsymbol{A} = \boldsymbol{\nabla}\chi$. Under such gauge transformation the phase changes to a different value φ', defined by $\varphi' = \varphi - \frac{e^*}{\hbar}\chi$. Therefore, the phase difference becomes

$$\Delta\varphi' = \int_1^2 \boldsymbol{\nabla}\varphi d\boldsymbol{l} - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{\nabla}\chi d\boldsymbol{l} = \Delta\varphi - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}d\boldsymbol{l}$$

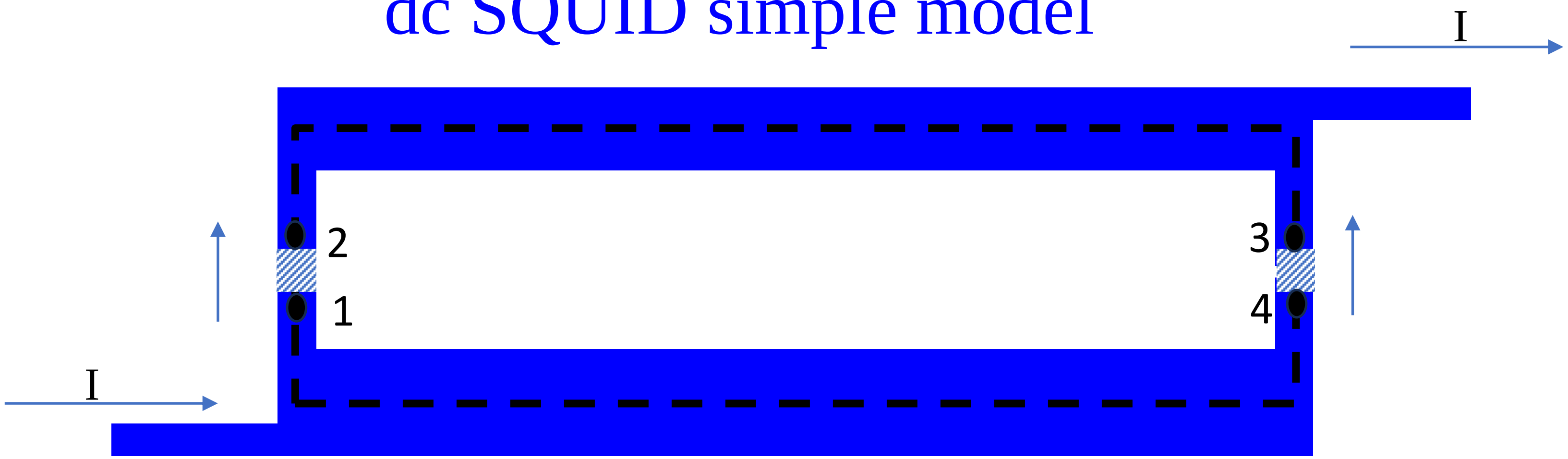
But the supercurrent must not change since all physical quantities are invariant under gauge transformations.
Thus, we introduce a gauge invariant phase difference such that it is equivalent to the phase difference at zero gauge but remains unchanged as we change the gauge. This gauge invariant phase difference is: $\Delta\gamma$.

$$\Delta\gamma = \Delta\varphi + \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}d\boldsymbol{l}$$

With such definition we achieve the required invariance $\Delta\gamma = \Delta\gamma'$ because

$$\Delta\gamma' = \Delta\varphi' + \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}d\boldsymbol{l} = \Delta\varphi - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}d\boldsymbol{l} + \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}d\boldsymbol{l} = \Delta\varphi = \Delta\gamma$$

dc SQUID simple model



Here the blue is bulk superconductor, white is empty space, dashed region is the insulator through which electrons transit by quantum tunnel.

The dashed black line is a contour over which we calculate the phase change. The phase accumulation on a closed contour like this one must be zero or 2πn, where n is an integer. This is required because the wavefunction must be a single-value function.

The total phase accumulation is $\Delta\varphi_{21} + \Delta\varphi_{32} + \Delta\varphi_{43} + \Delta\varphi_{14} = 2\pi n$

$$\Delta\varphi_{21} = \Delta\gamma_{21} - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}d\boldsymbol{l} \qquad \Delta\varphi_{43} = \Delta\gamma_{43} - \frac{e^*}{\hbar}\int_3^4 \boldsymbol{A}d\boldsymbol{l}$$

The integral of the vector potential over a circle is the magnetic flux through the SQUID loop:

$$\int_1^2 \boldsymbol{A}d\boldsymbol{l} + \int_1^2 \boldsymbol{A}d\boldsymbol{l} + \int_1^2 \boldsymbol{A}d\boldsymbol{l} + \int_1^2 \boldsymbol{A}d\boldsymbol{l} = \Phi$$

Magnetic flux definition: $\Phi = A_{loop}B$. Here A_{loop} is the area of the loop (the dashed line contour) and B is the magnetic field applied perpendicular to the loop.

dc SQUID simple model

$$\Delta\gamma_{21} - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}\boldsymbol{dl} + \Delta\varphi_{32} + \Delta\gamma_{43} - \frac{e^*}{\hbar}\int_3^4 \boldsymbol{A}\boldsymbol{dl} + \Delta\varphi_{14} = 2\pi n$$

$$\Delta\varphi_{21} = \Delta\gamma_{21} - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}\boldsymbol{dl}$$

$$\Delta\varphi_{43} = \Delta\gamma_{43} - \frac{e^*}{\hbar}\int_3^4 \boldsymbol{A}\boldsymbol{dl}$$

Remember the expression for the supercurrent: $\boldsymbol{j} = en_s\boldsymbol{v} = (e^*n_s/m^*)\,(\hbar\boldsymbol{\nabla}\varphi + e^*\boldsymbol{A})$

But in bulk superconductor the supercurrent is zero due to Meissner effect: $\boldsymbol{j} = 0$

Therefore: $\hbar\boldsymbol{\nabla}\varphi = -e^*\boldsymbol{A}$

Therefore, the phase accumulation in the superconductor bulk is:

$$\Delta\varphi_{32} = \int_2^3 \boldsymbol{\nabla}\varphi\boldsymbol{dl} = -\left(\frac{e^*}{\hbar}\right)\int_2^3 \boldsymbol{A}\boldsymbol{dl} \qquad \Delta\varphi_{14} = \int_4^1 \boldsymbol{\nabla}\varphi\boldsymbol{dl} = -\left(\frac{e^*}{\hbar}\right)\int_4^1 \boldsymbol{A}\boldsymbol{dl}$$

$$\Delta\gamma_{21} - \frac{e^*}{\hbar}\int_1^2 \boldsymbol{A}\boldsymbol{dl} - \left(\frac{e^*}{\hbar}\right)\int_2^3 \boldsymbol{A}\boldsymbol{dl} + \Delta\gamma_{43} - \frac{e^*}{\hbar}\int_3^4 \boldsymbol{A}\boldsymbol{dl} - \left(\frac{e^*}{\hbar}\right)\int_4^1 \boldsymbol{A}\boldsymbol{dl} = 2\pi n$$

The integral of the vector potential over a circle is the flux through the SQUID loop:

$$\int_1^2 \boldsymbol{A}\boldsymbol{dl} + \int_2^3 \boldsymbol{A}\boldsymbol{dl} + \int_3^4 \boldsymbol{A}\boldsymbol{dl} + \int_4^1 \boldsymbol{A}\boldsymbol{dl} = \Phi$$

Final result for SQUID:

$$\Delta\gamma_{21} + \Delta\gamma_{43} = \frac{e^*}{\hbar}\Phi + 2\pi n = \frac{e^*}{\hbar}\Phi + 2\pi n = 2\pi\frac{\Phi}{\phi_0} + 2\pi n$$

Usually, n=0 to minimize energy.

Here $\phi_0 = \frac{h}{e^*}$ is the magnetic flux quantum, which is a constant.

dc SQUID critical current

Now we can compute the total supercurrent trough the SQUID, using the formula $\Delta\gamma_{21} + \Delta\gamma_{43} = 2\pi\Phi/\phi_0 + 2\pi n$

$$I = I_c \sin\Delta\gamma_{21} + I_c \sin\Delta\gamma_{34} = I_c \sin\Delta\gamma_{21} - I_c \sin(2\pi\Phi/\phi_0 + 2\pi n - \Delta\gamma_{21}) =$$

$$\Delta\gamma_{43} = -\Delta\gamma_{34}$$

$$I_c \sin\Delta\gamma_{21} - I_c \sin(2\pi\Phi/\phi_0 - \Delta\gamma_{21}) =$$

$$I_c \sin\Delta\gamma_{21} + I_c \sin(\Delta\gamma_{21} - 2\pi\Phi/\phi_0) =$$

$$I_c[\sin\Delta\gamma_{21} + \sin(\Delta\gamma_{21} - 2\pi\Phi/\phi_0)] = 2I_c \cos(\frac{\pi\Phi}{\phi_0}) \sin\left(\Delta\gamma_{21} - \frac{\pi\Phi}{\phi_0}\right)$$

To find the critical current one needs to find the maximum possible value of the supercurrent for all possible choices of the phase difference $\Delta\gamma_{21}$.

The **critical current of the entire SQUID, as a function of magnetic flux** is: $I_{c,SQUID} = 2I_c \left|\cos\frac{\pi\Phi}{\phi_0}\right|$

Note that the critical current is, by definition, always positive. It is the maximum current which SQUID can sustain with zero voltage applied to it.