1) Buckyball spectrum.: Consider the symmetry group of the C_{60} buckyball molecule illustrated on page 194 of the notes.

a) Starting from the character table of the orientation-preserving icosohedral group Y (table 5.3), and using the fact that the $\mathbb{Z}_2$ parity inversion $\sigma : \mathbf{r} \to -\mathbf{r}$ combines with $g \in Y$ so that $D^{J_g}(\sigma g) = D^{J_g}(g)$, whilst $D^{J_u}(\sigma g) = -D^{J_u}(g)$, write down the character table of the extended group $Y_h = Y \times \mathbb{Z}_2$ that acts as a symmetry on the C_{60} molecule. There are now ten conjugacy classes, and the ten representations will be labelled A_g, A_u, *etc*. Verify that your character table has the expected row-orthogonality properties.

b) By counting the number of atoms left fixed by each group operation, compute the compound character of the action of Y_h on the C_{60} molecule. (Hint: Examine the pattern of panels on a regulation soccer ball, and deduce that four carbon atoms are left unmoved by operations in the class σC_2.)

c) Use your compound character from part b), to show that the 60-dimensional Hillbert space decomposes as

$$\mathcal{H}_{C_{60}} = A_g \oplus T_{1g} \oplus 2T_{1u} \oplus T_{2g} \oplus 2T_{2u} \oplus 2G_g \oplus 2G_u \oplus 3H_g \oplus 2H_u,$$

consistent with the energy-levels sketched in figure 5.3.

2) Matrix commutators:

a) Let $\hat\lambda_1$ and $\hat\lambda_2$ be hermitian matrices. Show that if we define $\hat\lambda_3$ by the relation $[\hat\lambda_1, \hat\lambda_2] = i\hat\lambda_3$, then $\hat\lambda_3$ is also a hermitian matrix.

b) For the Lie group $O(n)$, the matrices "$i\hat\lambda$" are real n-by-n skew symmetric matrices. Show that if A_1 and A_2 are real skew symmetric matrices, then so is $[A_1, A_2]$.

c) For the Lie group $Sp(2n, \mathbb{R})$, the $i\hat\lambda$ matrices are of the form

$$A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

where a is a real n-by-n matrix and b and c are symmetric ($a^T = a$ and $b^T = b$) real n-by-n matrices. Show that the commutator of any two matrices of this form is also of this form.

3 Euler angles and SU(2): Parametrize the elements of SU(2) as

$$\begin{aligned} U &= \exp\{-i\phi\hat\sigma_3/2\}\exp\{-i\theta\hat\sigma_2/2\}\exp\{-i\psi\hat\sigma_3/2\}, \\ &= \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}, \\ &= \begin{pmatrix} e^{-i(\phi+\psi)/2}\cos\theta/2 & -e^{i(\psi-\phi)/2}\sin\theta/2 \\ e^{i(\phi-\psi)/2}\sin\theta/2 & e^{+i(\psi+\phi)/2}\cos\theta/2 \end{pmatrix}. \end{aligned}$$

a) Show that Hopf : $S^3 \to S^2$ is the projection of $S^3 \simeq \mathrm{SU}(2)$ onto the coset space $S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$, where U(1) is the subgroup $\{\exp(-i\psi\hat{\sigma}_3/2)\}$.. Concude that Hopf takes $(\theta, \phi, \psi) \to (\theta, \phi)$, where θ and ϕ are spherical polar co-ordinates on the two-sphere.

b) Show that

$$U^{-1}dU = -\frac{i}{2}\hat{\sigma}_i\,\Omega^i_{\mathrm{L}},$$

where

$$\begin{aligned}
\Omega^1_{\mathrm{L}} &= \sin\psi\,d\theta - \sin\theta\cos\psi\,d\phi,\\
\Omega^2_{\mathrm{L}} &= \cos\psi\,d\theta + \sin\theta\sin\psi\,d\phi,\\
\Omega^3_{\mathrm{L}} &= d\psi + \cos\theta\,d\phi.
\end{aligned}$$

Compare these 1-forms with the components

$$\begin{aligned}
\omega_X &= \sin\psi\,\dot\theta - \sin\theta\cos\psi\,\dot\phi,\\
\omega_Y &= \cos\psi\,\dot\theta - \sin\theta\sin\psi\,\dot\phi,\\
\omega_Z &= \dot\psi + \cos\theta\,\dot\phi.
\end{aligned}$$

of the angular velocity $\boldsymbol{\omega}$ of a body with respect to the *body-fixed* XYZ.

c) (Optional) Now show that

$$dUU^{-1} = -\frac{i}{2}\hat{\sigma}_i\,\Omega^i_{\mathrm{R}},$$

where

$$\begin{aligned}
\Omega^1_{\mathrm{R}} &= -\sin\phi\,d\theta + \sin\theta\cos\psi\,d\psi,\\
\Omega^2_{\mathrm{R}} &= \cos\phi\,d\theta + \sin\theta\sin\psi\,d\psi,\\
\Omega^3_{\mathrm{R}} &= d\phi + \cos\theta\,d\psi,
\end{aligned}$$

Compare these 1-forms with components ω_x, ω_y, ω_z of the same angular velocity vector $\boldsymbol{\omega}$, but now with respect to the *space-fixed* xyz frame.

4) Class and group volume:

a) In the lecture notes I claimed that the volume fraction of the group SU(2) occupied by rotations through angles lying between θ and $\theta + d\theta$ is $\sin^2(\theta/2)d\theta/\pi$. By considering the geometry of the three-sphere, show that this is correct.

b) Show that

$$\int_{\mathrm{SU}(2)} \mathrm{tr}\,[(U^{-1}dU)^3] = 24\pi^2.$$

c) Suppose we have a map $g : \mathbb{R}^3 \to \mathrm{SU}(2)$ such that $g(x)$ goes to the identity element at infinity. Consider the integral

$$S[g] = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \mathrm{tr}\,[(g^{-1}dg)^3],$$

where the 3-form $\mathrm{tr}\,(g^{-1}dg)^3$ is the pull-back to $\mathbb{R}^3$ of the form $\mathrm{tr}\,[(U^{-1}dU)^3]$ on SU(2). Show that if we vary $g \to g + \delta g$, then

$$\delta S[g] = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} d\left\{3\,\mathrm{tr}\,[(g^{-1}\delta g)(g^{-1}dg)^2]\right\} = 0,$$

and so $S[g]$ is topological invariant of the map g. Conclude that the functional $S[g]$ is an integer, that integer being the Brouwer degree, or winding number, of the map $g : S^3 \to S^3$.

5) Campbell-Baker-Hausdorff Formulae: Here are some useful formula for working with exponentials of matrices that do not commute with each other.

a) Let X and X be matrices. Show that

$$e^{tX} Y e^{-tX} = Y + t[X, Y] + \frac{1}{2} t^2 [X, [X, Y]] + \cdots,$$

the terms on the right being the series expansion of $\exp[\mathrm{ad}(tX)]Y$. A proof is sketched in a footnote in the lecture notes, but I want you to fill in the details.

b) Let X and δX be matrices. Show that

$$\begin{aligned} e^{-X} e^{X+\delta X} &= 1 + \int_0^1 e^{-tX} \delta X e^{tX} dt + O\left[(\delta X)^2\right] \\ &= 1 + \delta X - \frac{1}{2}[X, \delta X] + \frac{1}{3!}[X, [X, \delta X]] + \cdots \\ &= 1 + \left(\frac{1 - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right) \delta X + O\left[(\delta X)^2\right] \end{aligned}$$

c) By expanding out the exponentials, show that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\text{higher}},$$

where "higher" means terms higher order in X, Y. The next two terms are, in fact, $\frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]]$.

6) SU(3): Here are some simple results that come from playing with the Gell-Mann lambda matrices, as well as practice at decomposing tensor products.

The totally antisymmetric structure constants, f_{ijk}, and a set of totally symmetric constants d_{ijk} are defined by

$$f_{ijk} = \frac{1}{2}\mathrm{tr}\,(\lambda_i[\lambda_j, \lambda_k]), \qquad d_{ijk} = \frac{1}{2}\mathrm{tr}\,(\lambda_i\{\lambda_j, \lambda_k\}).$$

Let $D^8_{ij}(g)$ be the matrices representing SU(3) in "8" — the eight-dimensional adjoint representation.

a) Show that

$$\begin{aligned} f_{ijk} &= D^8_{il}(g)D^8_{jm}(g)D^8_{kn}(g)f_{lmn}, \\ d_{ijk} &= D^8_{il}(g)D^8_{jm}(g)D^8_{kn}(g)d_{lmn}, \end{aligned} \tag{1}$$

and so f_{ijk} and d_{ijk} are *invariant tensors* in the same sense that δ_{ij} and $\epsilon_{i_1 \dots i_n}$ are invariant tensors for SO(n).

b) Let $w_i = f_{ijk}u_j v_k$. Show that if $u_i \to D^8_{ij}(g)u_j$ and $v_i \to D^8_{ij}(g)v_j$, then $w_i \to D^8_{ij}(g)w_j$. Similarly for $w_i = d_{ijk}u_j v_k$. (Hint: show first that the D^8 matrices are real and orthogonal.) Deduce that f_{ijk} and d_{ijk} are *Clebsh-Gordan coefficients* for the $8 \oplus 8$ part of the decomposition

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27.$$

a) Similarly show that $\delta_{\alpha\beta}$ and the lambda matrices $(\lambda_i)_{\alpha\beta}$ can be regarded as Clebsch-Gordan coefficients for the decomposition

$$3 \otimes \bar{3} = 1 \oplus 8.$$

d) Use the graphical method, introduced in class, of plotting weights and pealing off irreps to obtain the tensor product decomposition in part b).