1) Index Gymnastics and Einstein Convention: Using $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$ and $\mathbf{e}_\mu \cdot \mathbf{x} = x_\mu$, and also the definition of $\delta_{\mu\nu}$, verify the following statements for $\mathbb{R}^3$:

i) $\mathbf{x} \cdot \mathbf{x} = x_\mu x_\mu$
ii) $\mathbf{x} \cdot \mathbf{y} = x_\mu y_\mu$
iii) $\delta_{\mu\nu}\,\delta_{\nu\rho} = \delta_{\mu\rho}$
iv) $a_\mu = a_\nu \delta_{\nu\mu}$
v) $\delta_{\mu\mu} = 3$.

Still in $\mathbb{R}^3$ where the indices μ, ν take values 1, 2 and 3, and by writing out all the terms explicitly:

a) Show that
$$(A^\mu B_\mu)(C^\nu D_\nu) = (A^\mu C^\nu)(B_\mu D_\nu).$$
b) If $A_{\mu\nu} = -A_{\nu\mu}$ and $B^{\mu\nu} = B^{\nu\mu}$, show that
$$A_{\mu\nu} B^{\mu\nu} = 0.$$

Repeat exercise b) without writing out the terms explicitly, but instead by relabelling one or more of the dummy indices.

2) Antisymmetry: We define the n-dimensional Levi-Civita symbol by requiring that $\epsilon_{i_1 i_2 \ldots i_n}$ be antisymmetric in all pairs of indices, and $\epsilon_{12\ldots n} = 1$.

a) Show that $\epsilon_{123} = \epsilon_{231} = \epsilon_{312}$, but that $\epsilon_{1234} = -\epsilon_{2341}$, *etc.*
b) Show that
$$\epsilon_{ijk}\epsilon_{i'j'k'} = \delta_{ii'}\delta_{jj'}\delta_{kk'} + \text{five other terms},$$
where you should write out all six terms explicitly.
c) Show that $\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$.
d) For $n = 4$ write out $\epsilon_{ijkl}\epsilon_{ij'k'l'}$ as a sum of products of δ's similar to the one in part c)

3) Vector Products: The vector product of two three-vectors may be written in Cartesian components as $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$. Use this and your results about ϵ_{ijk} from the previous exercise to show that

i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$,
ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$,
iii) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.
iv) If $\mathbf{a}$ $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d} = \mathbf{b}$ are unit vectors, show that the identities i) and iii) are the sine and cosine rule, respectively, of spherical trigonometry. (Hint: for the spherical sine rule, begin by showing that $\mathbf{a} \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c})] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.)

4) Bernoulli and Vector Products: Write out Euler's equation for fluid motion

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla h$$

in Cartesian index notation. Use the properties of the Levi-Civita symbol to transform it into

$$\dot{\mathbf{v}} - \mathbf{v} \times \boldsymbol{\omega} = -\nabla \left(\frac{1}{2}\mathbf{v}^2 + h \right),$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity. Deduce Bernoulli's theorem, that for steady ($\dot{\mathbf{v}} = 0$) flow the quantity $\frac{1}{2}\mathbf{v}^2 + h$ is constant along streamlines.

5) Antisymmetry and Determinants: First the usual elementary definition, and then a slick abstract one.

a) Define the determinant of an n-by-n matrix A_{ij} by the expression

$$\det \mathbf{A} = \epsilon_{i_1 i_2 \ldots i_n} A_{1i_1} A_{2i_2} \ldots A_{ni_n}. \tag{1}$$

Without assuming any other knowledge you might have about properties of determinants, deduce from this definition that

$$\epsilon_{i_1 i_2 \ldots i_n} \det \mathbf{A} = \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n} \tag{2}$$

(hint: relabel the dummy indices) From this, deduce that $\det \mathbf{A} \det \mathbf{B} = \det(\mathbf{AB})$ where $\mathbf{AB}$ is the matrix with entries $A_{ik}B_{kj}$.

b) Let V be an n dimensional vector space over $\mathbb{C}$ (the complex numbers). Let $\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ be a skew-symmetric n-linear form, *i.e.* a map $\omega : \overbrace{V \times V \times \ldots V}^{n\,\text{factors}} \to \mathbb{C}$ which is linear in each slot, and changes sign when any two arguments are interchanged.

i) Show that there is only one such form up to a multiplicative constant.

ii) Assuming that ω does not vanish identically, show that a set of n vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is linearly independent if, and only if, $\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \neq 0$.

Now we define the determinant of the linear map $\mathbf{A} : V \to V$ by

$$(\det \mathbf{A})\, \omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \omega(\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{x}_n). \tag{3}$$

Show that this definition coincides with the one above. Use this new definition to again prove that $\det \mathbf{A} \det \mathbf{B} = \det(\mathbf{AB})$ where $\mathbf{AB}$ is now interpreted as the composition of the two maps. Notice that, although there is rather more overhead in setting up the new definition, this proof is much more transparent than the old!