Physics 509 Homework 3

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1 Infinitesimal Homotopy

The infinitesimal homotopy relation states

$$\mathcal{L}_X \omega = (d i_X + i_X d)\omega. \tag{1}$$

Taking the exterior derivative of (1),

$$\begin{aligned} d(\mathcal{L}_X \omega) &= d(d i_X + i_X d)\omega && \text{(using (1))} \\ &= \cancel{d^2 i_X \omega} + d(i_X d\omega) && (d^2 = 0) \\ &= (d i_X + i_X d)(d\omega) && (\text{add } i_X d^2\omega = 0) \\ &= \mathcal{L}_X (d\omega). \end{aligned}$$

2 Magnetic Solid

(a) We need to verify that

$$\dot{\mathbf{x}} = \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \mathbf{\Omega} \tag{2a}$$

$$\dot{\mathbf{k}} = -\frac{\partial V}{\partial \mathbf{x}} - e\dot{\mathbf{x}} \times \mathbf{B} \tag{2b}$$

is indeed the Hamiltonian vector flow of $H(\mathbf{x}, \mathbf{k}) = \epsilon(\mathbf{k}) + V(\mathbf{x})$ with the symplectic form ω. This amounts to checking that

$$dH = -i_{v_H}\omega = -\omega(v_H, \,\cdot\,) \tag{3}$$

reproduces equations (2b) and (2b). First, expand dH in local coordinates we find

$$dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial k^i} dk^i = \frac{\partial V(\mathbf{x})}{\partial x^i} dx^i + \frac{\partial \epsilon(\mathbf{k})}{\partial k^i} dk^i. \tag{4}$$

Next, plugging in the velocity vector field,

$$v_H = \dot{x}^i \frac{\partial}{\partial x^i} + \dot{k}^i \frac{\partial}{\partial k^i}, \tag{5}$$

into $-\omega(v_H,\ \cdot\ )$, we find[1]

$$
\begin{aligned}
-\omega(v_H,\ \cdot\ ) &= -\left\{dk^i dx^i - \frac{e}{2}\epsilon_{ijk}B^i(\mathbf{x})dx^j dx^k + \frac{1}{2}\epsilon_{ijk}\Omega^i(\mathbf{k})dk^j dk^k\right\}\left(\dot{x}^\ell \frac{\partial}{\partial x^\ell} + \dot{k}^\ell \frac{\partial}{\partial k^\ell},\ \cdot\ \right) \\
&= -\dot{k}_i dx^i + \dot{x}_i dk^i + \frac{e}{2}\epsilon_{ijk}B^i(\mathbf{x})\left[\dot{x}_j dx^k - \dot{x}_k dx^j\right] - \frac{1}{2}\epsilon_{ijk}\Omega^i(\mathbf{k})\left[\dot{k}_j dk^k - \dot{k}_k dk^j\right] \\
&= \Big[\underbrace{-\dot{k}_k - e\epsilon_{ijk}\dot{x}^i B^j(\mathbf{x})}_{=\frac{\partial V(\mathbf{x})}{\partial x^k}}\Big]dx^k + \Big[\underbrace{\dot{x}_k + \epsilon_{ijk}\dot{k}^i\Omega^j(\mathbf{k})}_{=\frac{\partial \epsilon(\mathbf{k})}{\partial k^k}}\Big]dk^k,
\end{aligned}
$$

where in the last line I've relabeled indices, $i \leftrightarrow j$, and used equation (4) to write the underset equalities. These equalities reproduce equations (2a) and (2b), as desired.

(b) We first check that ω is closed.

$$
\begin{aligned}
d\omega &= d\left\{dk^i dx^i - \frac{e}{2}\epsilon_{ijk}B^i(\mathbf{x})dx^j dx^k + \frac{1}{2}\epsilon_{ijk}\Omega^i(\mathbf{k})dk^j dk^k\right\} \\
&= -\frac{e}{2}\epsilon_{ijk}\left(dB^i(\mathbf{x})\right)dx^j dx^k + \frac{1}{2}\epsilon_{ijk}\left(d\Omega^i(\mathbf{k})\right)dk^j dk^k \\
&= -\frac{e}{2}\epsilon_{ijk}\left(\frac{\partial B^i(\mathbf{x})}{\partial x^\ell}dx^\ell\right)dx^j dx^k + \frac{1}{2}\epsilon_{ijk}\left(\frac{\Omega^i(\mathbf{k})}{\partial k^\ell}dk^\ell\right)dk^j dk^k \\
&= -\frac{e}{2}\epsilon_{ijk}\left(\frac{\partial B^i(\mathbf{x})}{\partial x^i}\right)dx^i dx^j dx^k + \frac{1}{2}\epsilon_{ijk}\left(\frac{\Omega^i(\mathbf{k})}{\partial k^i}\right)dk^i dk^j dk^k \quad (\text{antisymmetry} \implies \ell = i).
\end{aligned}
$$

Now the product $\epsilon_{ijk}dx^i dx^j dx^k$ is just proportional to $dx^1 dx^2 dx^3$ (and likewise for the $dk^i dk^j dk^k$). Hence we can write

$$
d\omega \propto -\frac{e}{2}\underbrace{\left(\frac{\partial B^i(\mathbf{x})}{\partial x^i}\right)}_{\mathrm{div}_{\mathbf{x}}\,\mathbf{B}}dx^1 dx^2 dx^3 + \frac{1}{2}\underbrace{\left(\frac{\Omega^i(\mathbf{k})}{\partial k^i}\right)}_{\mathrm{div}_{\mathbf{k}}\,\mathbf{\Omega}}dk^1 dk^2 dk^3.
$$

But this vanishes identically since $\mathrm{div}_{\mathbf{x}}\,\mathbf{B} = \mathrm{div}_{\mathbf{k}}\,\mathbf{\Omega} = 0$. Hence $d\omega = 0$, as desired.

To show the desired Poisson brackets, first we find expressions for $\dot{x}_i$ and $\dot{k}_i$ using equations (2a) and (2b). To this end, note the following dot product equalities,

$$
\text{(2a)} \implies \dot{\mathbf{x}} \cdot \mathbf{\Omega} = \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \cdot \mathbf{\Omega} - \cancelto{=0}{(\dot{\mathbf{k}} \times \mathbf{\Omega}) \cdot \mathbf{\Omega}} \tag{6a}
$$

$$
\text{(2b)} \implies \dot{\mathbf{k}} \cdot \mathbf{B} = -\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{B} - \cancelto{=0}{(e\dot{\mathbf{x}} \times \mathbf{B}) \cdot \mathbf{B}}. \tag{6b}
$$

[1] For notational convenience, I will drop the wedge product and write $dx^i dx^j$ in place of $dx^i \wedge dx^j$. I will also simply write dx^i rather than $dx^i(\ \cdot\ )$, although it is still implied that the dual basis elements act on basis elements (of the tangent space).

Plugging (2b) into (2a),

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} - \left[-\frac{\partial V(\mathbf{x})}{\partial x} - e\dot{\mathbf{x}} \times \mathbf{B}\right] \times \boldsymbol{\Omega} \\
&= \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} + \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \times \boldsymbol{\Omega} - [e\dot{\mathbf{x}}(\boldsymbol{\Omega} \cdot \mathbf{B}) - \mathbf{B}(e\dot{\mathbf{x}} \cdot \boldsymbol{\Omega})] && \text{(BAC-CAB Rule)} \\
&= \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} + \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \times \boldsymbol{\Omega} - \left[e\dot{\mathbf{x}}(\boldsymbol{\Omega} \cdot \mathbf{B}) - \mathbf{B}(e\frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \cdot \boldsymbol{\Omega})\right] && \text{(equation (6a))}.
\end{aligned}$$

This rearranges to

$$\dot{\mathbf{x}}(1 + e\mathbf{B} \cdot \boldsymbol{\Omega}) = \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} + \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \times \boldsymbol{\Omega} + \mathbf{B}\left(e\frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \cdot \boldsymbol{\Omega}\right). \tag{7}$$

The analogous procedure for $\dot{\mathbf{k}}$ yields

$$\dot{\mathbf{k}}(1 + e\mathbf{B} \cdot \Omega) = -\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} - e\frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \times \mathbf{B} - \boldsymbol{\Omega}\left(e\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{B}\right). \tag{8}$$

Now, in part (a) we showed that equations (2a) and (2b) are Hamiltonian with ω as the symplectic form for any Hamiltonian of the form $H(\mathbf{x}, \mathbf{k}) = \epsilon(\mathbf{k}) + V(\mathbf{x})$. We can then easily relate the time derivatives of functions with the Poisson bracket with a Hamiltonian function via[2]

$$\{H_1, H_2\} \overset{\text{def}}{=} \left.\frac{dH_2}{dt}\right|_{H_1} = \dot{H}_2. \tag{9}$$

Or equivalently, $\{f, H\} = -\dot{f}$ (since $\{f, g\} = -\{g, f\}$).

The computation of the Poisson brackets follows immediately. Choosing $f = x_i$ and $H = x_j$, equation (7) yields

$$\{x_i, x_j\} = -\dot{x}_i = -\frac{\epsilon_{ijk}\Omega_k}{(1 + e\mathbf{B} \cdot \Omega)}.$$

The remaining two Poisson brackets follow by the same procedure but with the Hamiltonian function $H = k_j$. Summarizing, one finds

$$\{x_i, x_j\} = -\frac{\epsilon_{ijk}\Omega_k}{(1 + e\mathbf{B} \cdot \boldsymbol{\Omega})}, \qquad \{x_i, k_j\} = -\frac{\delta_{ij} + eB_i\Omega_j}{(1 + e\mathbf{B} \cdot \boldsymbol{\Omega})}, \qquad \{k_i, k_j\} = \frac{\epsilon_{ijk}eB_k}{(1 + e\mathbf{B} \cdot \boldsymbol{\Omega})}.$$

(c) The conserved phase-space volume $\omega^3/3!$ can be computed by direct calculation. Note that terms like $dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell$ vanish since necessarily there will be one repeated index for

[2] See equation (11.96) in the textbook. Also note that the definition given here and in the textbook differs by a minus sign from the traditional one. The literature is sometimes inconsistent with which definition is used, so it is always worth checking the convention used.

three spatial dimensions (an analogous argument holds for the k's). Hence

$$\begin{aligned}\omega^3 &= dx^i dk^i dx^j dk^j dx^k dk^k + 3!(dk^i dx^i)\left(-\frac{e}{2}\epsilon_{i'j'k'}B^{i'}dx^{j'}dx^{k'}\right)\left(\frac{1}{2}\epsilon_{i''j''k''}\Omega^{i''}dk^{j''}dk^{k''}\right)\\ &= dk^i dk^j dk^k dx^i dx^j dk^k - 3!\frac{e}{4}\left(\epsilon_{i'j'k'}\epsilon_{i''j''k''}B^{i'}\Omega^{i''}\right)dk^i dx^i dx^{j'}dx^{k'}dk^{j''}dk^{k''}\\ &= 3!\left[1+(e\mathbf{B}\cdot\mathbf{\Omega})\right]dk^1dk^2dk^3dx^1dx^2dx^3,\end{aligned}$$

which implies $\omega^3/3! = (1+e\mathbf{B}\cdot\mathbf{\Omega})d^3k d^3x$, as desired.

3 Non-abelian Gauge Fields as Matrix-valued Forms

(i) Given $A = A_\mu dx^\mu$, write

$$2A^2 = A_\mu A_\nu dx^\mu dx^\nu + A_\mu A_\nu dx^\mu dx^\nu = (A_\mu A_\nu - A_\nu A_\mu)dx^\mu dx^\nu$$
$$\implies A^2 = \frac{1}{2}[A_\mu, A_\nu]dx^\mu dx^\nu,$$

where in the last equality I've used $dx^\mu dx^\nu = -dx^\nu dx^\mu$ and relabeled indices. Similarly, one finds $dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu dx^\nu$, so that

$$F = dA + A^2 = \underbrace{(\partial_\mu\partial_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])}_{=F_{\mu\nu}}dx^\mu dx^\nu.$$

(ii) Using the definition of the gauge-covariant derivatives,

$$\nabla_\mu = \partial_\mu - A_\mu,$$

one finds

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] &= \nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\\ &= (\partial_\mu - A_\mu)(\partial_\nu - A_\nu) - (\partial_\nu - A_\nu)(\partial_\mu - A_\mu)\\ &= \partial_\mu\partial_\nu + (\partial_\mu A_\nu) + A_\nu\partial_\mu + A_\mu\partial_\nu + A_\mu A_\nu && \text{(expand and cancel)}\\ &\quad - \partial_\nu\partial_\mu - (\partial_\nu A_\mu) - A_\mu\partial_\nu - A_\nu\partial_\mu - A_\nu A_\mu\\ &= (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + A_\mu A_\nu - A_\nu A_\mu\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\\ &= F_{\mu\nu} && \text{(by part (i)).}\end{aligned}$$

(iii) Let g be an invertable matrix and δg be a matrix describing a small change in g (we assume

$g + \delta g$ is still invertable).

$$(g+\delta g)(g^{-1}+\delta(g^{-1})) = \text{id} \implies \underbrace{gg^{-1}}_{=\text{id}} + g(\delta g)^{-1} + (\delta g)g^{-1} + \underbrace{(\delta g)\delta(g^{-1})}_{\mathcal{O}(\delta^2)} = \text{id}$$
$$\implies (\delta g)^{-1} = -g^{-1}(\delta g)g^{-1}.$$

Alternatively, by demanding there is no variation in the identity, we find $0 = \delta(gg^{-1}) = (\delta g)g^{-1} + g\delta(g^{-1}) \implies \delta(g^{-1}) \equiv (\delta g)^{-1} = -g^{-1}(\delta g)g^{-1}$.

(iv) Suppose that the matrix-valued gauge field is a "pure gauge"; i.e., that $A = g^{-1}dg$. Then

$$dA = d(g^{-1}dg) = d(g^{-1})dg = -g^{-1}dg\, g^{-1}dg = \left(g^{-1}dg\right)^{-1}.$$

This shows that

$$F = dA + A^2 = -\left(g^{-1}dg\right)^2 + \left(g^{-1}dg\right)^2 = 0,$$

as desired.

(v) Under a gauge transformation,

$$A_\mu \mapsto A_\mu^g \equiv g^{-1}A_\mu g + g^{-1}(\partial_\mu g).$$

Therefore, the covariant derivative transforms like

$$\nabla_\mu \mapsto \nabla_\mu^g \equiv \partial_\mu + A_\mu^g = g^{-1}g\partial_\mu + g^{-1}(\partial_\mu g) + g^{-1}A_\mu g = g^{-1}(\partial_\mu + A_\mu)g.$$

In the last equality, we have used the fact that the derivative acts to the right along with the chain rule (in reverse). Hence, $\nabla_\mu \mapsto g^{-1}\nabla_\mu g$ under a gauge transformation. Using the result from part (ii) ($F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$), we can easily find how $F_{\mu\nu}$ behaves when transformed.

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \mapsto [g^{-1}\nabla_\mu g, g^{-1}\nabla_\nu g] = g^{-1}[\nabla_\mu, \nabla_\nu]g = g^{-1}F_{\mu\nu}g,$$

as desired.

(vi) To show the Bianchi identity, we simply take the exterior derivative of F.

$$\begin{aligned} dF &= d(dA + A^2) \\ &= \cancel{d^2 A} + (dA)A - A(dA) && (d^2 = 0) \\ &= (F - A^2)A - A(F - A^2) && (dA = F - A^2) \\ &= FA - AF. \end{aligned}$$

This rearranges to $dF - FA + AF = 0$, as desired.

(vii) Next, use the Bianchi identity to show that the 4-form is closed.

$$\begin{aligned}
d\,\mathrm{tr}(F^2) &= \mathrm{tr}(dF^2) && (d\,\mathrm{tr} = \mathrm{tr}\,d)\\
&= \mathrm{tr}((dF)F + F(dF))\\
&= \mathrm{tr}((FA - AF)F + F(FA - AF)) && \text{(Bianchi identity)}\\
&= \mathrm{tr}(\cancel{FAF} - AFF + FFA - \cancel{FAF})\\
&= -\,\mathrm{tr}(AFF) + \mathrm{tr}(AFF) && \text{(cyclic perm.)}\\
&= 0.
\end{aligned}$$

Note that in this case the cyclic permutation of matrix-valued forms is also even so that this operation doesn't change the sign in the wedge product,[3]

$$\begin{aligned}
\mathrm{tr}(FFA) &= \mathrm{tr}(F_{\mu\nu}F_{\gamma\delta}A_\lambda)dx^\mu dx^\nu dx^\gamma dx^\delta dx^\lambda\\
&= (-1)^4\,\mathrm{tr}(A_\lambda F_{\mu\nu}F_{\gamma\delta})dx^\lambda dx^\mu dx^\nu dx^\gamma dx^\delta\\
&= \mathrm{tr}(AFF).
\end{aligned}$$

(viii) Before showing that

$$\mathrm{tr}(F^2) = d\left\{\mathrm{tr}\left(AdA + \frac{2}{3}\right)\right\}, \tag{10}$$

first note that $\mathrm{tr}(A^4) = 0$ since

$$\begin{aligned}
\mathrm{tr}(A^4) &= \mathrm{tr}(A_\mu A_\nu A_\gamma A_\delta)dx^\mu dx^\nu dx^\gamma dx^\delta\\
&= (-1)^3\,\mathrm{tr}(A_\delta A_\mu A_\nu A_\gamma)dx^\delta dx^\mu dx^\nu dx^\gamma\\
&= -\,\mathrm{tr}(A^4).
\end{aligned}$$

In the second equality, we have cyclically permuted matrices under the trace; however, unlike the product of three matrices in part (vii), this is obtained via an odd permutation which introduces a minus in the 4-form. With this, we can show (10) beginning from the right-hand-side.

$$\begin{aligned}
d\left\{\mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)\right\} &= \mathrm{tr}\left\{(dA)^2 + \frac{2}{3}\left[(dA)A^2 + \overbrace{A(dA)A}^{=\frac{1}{2}A^2(dA)+\frac{1}{2}(dA)A^2} + A^2(dA)\right]\right\}\\
&= \mathrm{tr}\left\{(dA)^2 + A^2(dA) + (dA)A^2 + (A^2)^2\right\} && (\mathrm{tr}(A^4) = 0)\\
&= \mathrm{tr}\left\{(dA + A^2)^2\right\}\\
&= \mathrm{tr}\left\{F^2\right\}.
\end{aligned}$$

Hence $\mathrm{tr}(F^2)$ is also exact. In the first line, I've used the cyclic property of the trace to write

[3] Again, the wedge product is implicit.

$A(dA)A$ in a symmetric way.