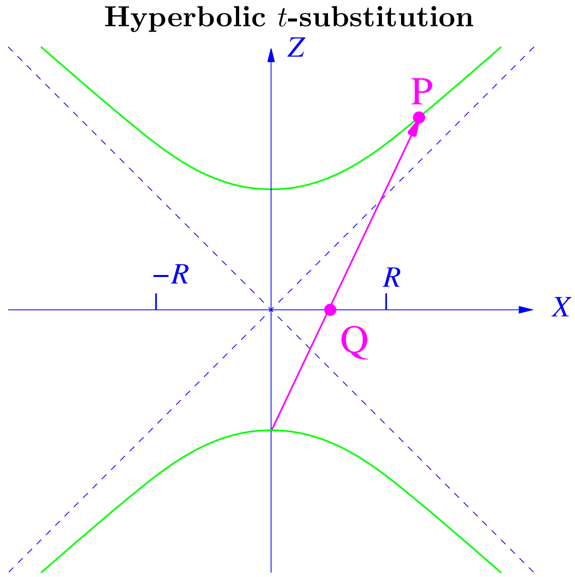
Physics 509 Homework 5 Professor M. Stone
Spring 2021 University of Illinois

1 Lobachevski Space

There are several ways to do this problem. Various possible solutions are shown below.

Method 1: (Mike's solution)



If we take $R = 1$ the point P has coordinates

$$X = \sinh s, \quad Z = \cosh s,$$

where, in the geometry of Lorentz boosts, s would be the *rapidity*. We can use the hyperbolic version

$$\sinh s = \frac{2t}{1-t^2}, \quad \cosh s = \frac{1+t^2}{1-t^2}$$

of the t-subsitution. This satisfies $\cosh^2 s - \sinh^2 s = 1$ as it should. The geometry of the figure, followed by a line of algebra, shows that the tangent of the angle between the line QP and the Z axis is

$$\frac{\sinh s}{1+\cosh s} = t.$$

Thus t has the geometric interpretation of being the radial distance in the X, Y plane from the origin to point Q.

The Minkowski arc length is

$$dX^2 - dZ^2 = (d\sinh s)^2 - (d\cosh s)^2 = (\cosh^2 s - \sinh^2 s)ds^2 = ds^2$$

so ds plays the role on the unit Minkowski hyperbola as $d\theta$ on the unit circle. From

$$\sinh s = \frac{2t}{1-t^2}$$

we read off that

$$(\cosh s)\, ds = \frac{2(1+t^2)}{(1-t^2)^2}\, dt$$

or

$$ds = \frac{2}{1-t^2}\, dt.$$

Thus for radial displacements

$$ds^2 = \frac{4}{1-t^2}\, dt^2 = \frac{4}{1-X^2+Y^2}\, dt^2 = \frac{4}{1-X^2+Y^2}(dX^2+dY^2).$$

As $dX^2 + dY^2 = dt^2 + t^2 d\phi^2$ and for angular displacements $ds^2 = \sinh^2 s\, d\phi^2$ the formula is correct in that case also.

Method 2: One can also follow analogous steps to what was done in the "Stereographic Projection" problem from the previous homework set. Using hyperbolic polar coordinates,

$$\begin{aligned} x(\theta,\phi) &= R\cos\phi\sinh\theta \\ y(\theta,\phi) &= R\sin\phi\sinh\theta \\ z(\theta,\phi) &= R\cosh\theta, \end{aligned}$$

the metric is $g(\ ,\ ) = d\phi\otimes d\phi + \sinh^2\theta\, d\theta\otimes d\theta$. This can then be mapped to the Poincaré disk model via the transformation $\zeta = X + iY = e^{i\phi}\tanh(\theta/2)$. Following identical steps to the computation performed in the previous homework (i.e., compute the Jacobian, then use it to transform g, which is just a doubly covariant tensor), one finds the new induced metric to be

$$\frac{4R^4}{\left(R^2 - X^2 - Y^2\right)^2}\left(dX\otimes dX + dY\otimes dY\right).$$

Note that here X and Y are coordinates on the Poincaré disk (in the previous problem set, the analogous variables were named ξ and η) whereas x, y, and z are coordinates on the upper half hyperboloid.

Method 3: Another "brute force" procedure one might follow is to start with the stereographic projection,

$$X(x,y) = R\left(\frac{2Rx}{R^2+x^2+y^2}\right)$$
$$Y(x,y) = R\left(\frac{2Ry}{R^2+x^2+y^2}\right)$$
$$Z(x,y) = R\left(\frac{-R^2+x^2+y^2}{R^2+x^2+y^2}\right),$$

where $\{X, Y, Z\}$ are the coordinates on S^2 and $\{x, y\}$ are the coordinates in the plane, and then plug in an imaginary radius (i.e., take $R \mapsto iR$ in the above mapping) as suggested in the problem. The induced metric is just that of the Poincaré disk model. The computation can be performed easily in Mathematica.

```
In[1]:= (* stereographic projection *)
        SetAttributes[R, Constant];
        X[x_, y_] := R (2 R x / (R^2 + (x^2 + y^2)));
        Y[x_, y_] := R (2 R y / (R^2 + (x^2 + y^2)));
        Z[x_, y_] := R ((-R^2 + (x^2 + y^2)) / (R^2 + (x^2 + y^2)));

        (* substitute in an "imaginary radius" and calculate metric *)
        Dt[X[x, y] /. R → I R]^2 + Dt[Y[x, y] /. R → I R]^2 + Dt[Z[x, y] /. R → I R]^2 //
         FullSimplify

Out[5]= 4 R^4 (Dt[x]^2 + Dt[y]^2) / (-R^2 + x^2 + y^2)^2
```

2 Flywheel and Rolling Ball

(a) Here we work in the *body-frame coordinates*, with the (principle) Z axis along the direction of the axle. In these coordinates, the inertia tensor is diagonal and, as a result of the symmetry about the axle, $I_{XX} = I_{YY}$. Since there are no external torques, we have that $L_Z = I_{ZZ}\omega_Z = I_{ZZ}(\dot{\psi} + \dot{\phi}\cos\theta)$ is a constant of motion.[1] When the axle has returned to rest in the initial position, we have $L_Z = 0$; hence, $\dot{\psi} = -\dot{\phi}\cos\theta$ at all points on the curve $\gamma = \partial\Omega$. Integrating this over the time required to make a closed loop, we find

$$\Delta\psi = -\int_0^\tau \dot{\phi}(t)\cos\theta(t)\,dt$$

[1] One can also see this via the Lagrangian and the Euler-Lagrange equations.

$$
\begin{aligned}
&= -\int_{\partial\Omega} \cos\theta(\phi)\, d\phi && \text{(parametrize } \theta \text{ in terms of } \phi\text{)} \\
&= -\int_{\Omega} d(\cos\theta\, d\phi) && \text{(Stokes' Theorem)} \\
&= \int_{\Omega} \sin\theta\, d\theta \wedge d\phi \\
&= \text{Area}(\Omega).
\end{aligned}
$$

Notice that if we reverse the orientation of the path, then the enclosed area becomes $4\pi - \text{Area}(\Omega)$. Since reversing orientation changes the sign, we have that $4\pi - \text{Area}(\Omega) = -\text{Area}(\Omega)$, which shows the area is only defined modulo 4π.

(b) Since the point in contact with the table describes a closed path on the ball, we instead use *space-fixed coordinates*[2] so that $\omega_Z = \dot{\phi} + \dot{\psi}\cos\theta$, and the no slip condition implies $\dot{\phi} + \dot{\psi}\cos\theta = 0$.[3] Analogous steps to those of part (a) show that $\Delta\phi = \text{Area}(\Omega)$.

3 Hopf Invariant

Before delving into calculations, it is worth summarizing some of the notation and identities we make use of throughout the solution. Given in the problem, we have

$$
\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\nabla P \qquad \text{(Euler's equation)} \tag{1}
$$

$$
\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \qquad \text{(incompressibility condition).} \tag{2}
$$

We also use the following vector calculus identities, which are easily proved by writing terms out in index notation.[4]

$$
\boldsymbol{\nabla}\cdot(\psi\mathbf{A}) = (\nabla\psi)\cdot\mathbf{A} + \psi(\boldsymbol{\nabla}\cdot\mathbf{A}) \tag{3}
$$

$$
\nabla(\mathbf{A}\cdot\mathbf{B}) = \nabla_{\mathbf{A}}(\mathbf{A}\cdot\mathbf{B}) + \nabla_{\mathbf{B}}(\mathbf{A}\cdot\mathbf{B}) \tag{4}
$$

$$
\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) = \nabla_{\mathbf{B}}(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} \tag{5}
$$

$$
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}, \tag{6}
$$

where I've used Feynman's subscript notation, $\nabla_{\mathbf{A}}(\mathbf{A}\cdot\mathbf{B}) \equiv B_k(\partial_j A_k)\hat{\mathbf{e}}_j$, to denote the gradient acts only on the vector in the subscript.

(a) (i) First note in equation (5), when $\mathbf{A} = \mathbf{B} = \mathbf{v}$, the $\nabla_{\mathbf{B}}(\mathbf{A}\cdot\mathbf{B})$ can be written as $\nabla\left(\frac{1}{2}\mathbf{v}^2\right)$. We can therefore write the curl of the convective derivative, $\nabla\times\frac{D\mathbf{v}}{Dt} = \frac{D}{Dt}(\nabla\times\mathbf{v}) = \frac{D\boldsymbol{\omega}}{Dt}$

[2] Note that the expression for the angular velocity vector differs in space-fixed and body-fixed coordinates (see here).

[3] This is the coordinate system employed in question 3 of homework 2 where we calculated the vector fields corresponding to the motions of a rolling ball.

[4] Wikipedia includes a comprehensive list, https://en.wikipedia.org/wiki/Vector_calculus_identities.

as

$$
\begin{aligned}
\frac{D\boldsymbol{\omega}}{Dt} &= \nabla \times \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right] \\
&= \frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{v}) + \nabla\times[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] \\
&= \frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla}\times\left[\nabla\left(\frac{1}{2}\mathbf{v}^2\right) - \mathbf{v}\times\boldsymbol{\omega}\right] \\
&= \frac{\partial\boldsymbol{\omega}}{\partial t} + \underbrace{\boldsymbol{\nabla}\times\left[\nabla\left(\frac{1}{2}\mathbf{v}^2\right)\right]}_{=0} - \boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega}) \qquad \text{(curl of gradient vanishes).}
\end{aligned}
$$

Now expanding the remaining term using (6), we find

$$
-\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega}) = \underbrace{-\mathbf{v}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega})}_{\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v})=0} + \overbrace{\boldsymbol{\omega}(\nabla\cdot\mathbf{v})}^{=0,\ \text{incompressible}} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}.
$$

Plugging this in, one finds

$$
\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v} = \underbrace{\boldsymbol{\nabla}\times(-\nabla P)}_{=0},
$$

which re-arranges to

$$
\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}, \tag{7}
$$

as desired.

(ii) Using the product rule and plugging in equations (1) and (7) we find

$$
\begin{aligned}
\frac{D}{Dt}(\mathbf{v}\cdot\boldsymbol{\omega}) &= \left[\frac{D\mathbf{v}}{Dt}\right]\cdot\boldsymbol{\omega} + \mathbf{v}\cdot\left[\frac{D\boldsymbol{\omega}}{Dt}\right] \\
&= [-\nabla P]\cdot\boldsymbol{\omega} + \mathbf{v}\cdot[(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}] \\
&= \boldsymbol{\omega}\cdot\left[-\nabla P + \nabla\left(\frac{1}{2}\mathbf{v}^2\right)\right] \qquad \left(\mathbf{v}\cdot[(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}] = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\left(\tfrac{1}{2}\mathbf{v}^2\right)\right) \\
&= \boldsymbol{\nabla}\cdot\left[\boldsymbol{\omega}\left(\frac{1}{2}\mathbf{v}^2 - P\right)\right] \qquad \text{(equation (3), } \boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0\text{).}
\end{aligned}
$$

(iii) For a volume $\Omega(t)$ that is co-moving with a fluid (and is allowed to change shape), we need some kind of generalization of Leibniz's integral rule. In the three dimensional case,[5] the appropriate generalization is known as the Reynolds Transport Theorem, which for

[5] Generalizations to higher dimensions can be clearly stated in the language of differential forms. See later comments for problem 4.

incompressible fluids takes the form

$$\frac{d}{dt}\int_{\Omega(t)} f(\mathbf{x},t)\,dV = \int_{\Omega(t)} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) f(\mathbf{x},t)\,dV. \tag{8}$$

This is the exact statement we are asked to check in the problem. To prove this, consider a parametrized family of diffeomorphisms, $\varphi_t : \Omega_0 \to \Omega(t)$ such that $\varphi_t : \mathbf{u} \mapsto \mathbf{x}$, which maps the region $\Omega_0 \equiv \Omega(t=0)$ to the corresponding region after it has been carried along the vector field $\mathbf{v}$ for some time t. Pulling back by this function, we can write

$$\int_{\Omega(t)} f \underbrace{dx^1 dx^2 dx^3}_{(=dV)} = \int_{\Omega_0} \varphi_t^*\,(f\,dV) = \int_{\Omega_0} f(\mathbf{x}(\mathbf{u}),t)\,|J|\,\underbrace{du^1 du^2 du^3}_{(=dV_0)},$$

where $|J| \equiv \left|\det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)\right|$.[6] This essentially moves the time-dependence of the region of integration into the integrand. We can then use Leibniz's rule to write

$$\frac{d}{dt}\int_{\Omega_0} f\,|J|\,dV_0 = \int_{\Omega_0} \left[\left(\frac{\partial}{\partial t} + \frac{\partial \mathbf{u}}{\partial t}\cdot\boldsymbol{\nabla}_{\mathbf{u}}\right) f\,|J| + f\left(\frac{\partial}{\partial t}\,|J|\right)\right] dV_0.$$

Note that $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and $\frac{\partial}{\partial t}\,|J| = 0$ since

$$\frac{d}{dt}\,\mathrm{Vol}(\Omega(t)) = \frac{d}{dt}\int_{\Omega(t)} dV = \frac{d}{dt}\int_{\Omega_0} |J|\,dV_0 = \int_{\Omega_0}\left(\frac{d}{dt}\,|J|\right) dV_0,$$

so that the incompressibility condition $\frac{d}{dt}\,\mathrm{Vol}(\Omega(t)) = 0$ implies that $\frac{d}{dt}\,|J| = 0$. After changing back to the original variables and placing the time-dependence back into the integration region, one finds

$$\frac{d}{dt}\int_{\Omega(t)} f(\mathbf{x},t)\,dV = \int_{\Omega(t)} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) f(\mathbf{x},t)\,dV = \int_{\Omega(t)} \frac{Df}{Dt}\,dV,$$

as desired.

(iv) Utilizing the results of the previous parts,

$$\begin{aligned}
\frac{d}{dt}H &= \frac{d}{dt}\int \mathbf{v}\cdot\boldsymbol{\omega}\,dV \\
&= \int \frac{D}{Dt}\,(\mathbf{v}\cdot\boldsymbol{\omega})\,dV && \text{(part (iii))} \\
&= \int \boldsymbol{\nabla}\cdot\left\{\boldsymbol{\omega}\left(\frac{1}{2} - P\right)\right\} dV && \text{(part (ii))} \\
&= \int \left\{\boldsymbol{\omega}\left(\frac{1}{2} - P\right)\right\}\cdot d\mathbf{S} && \text{(Gauss's law)}
\end{aligned}$$

[6]Notice that here we need the absolute values on the Jacobian because we are considering the unoriented integral. In the final question, we will perform a similar calculation in the language of differential forms were the integrals are *oriented*. There the Jacobian factor is included with no absolute values.

$$= 0 \qquad\qquad (||\boldsymbol{\omega}|| \to 0 \text{ at spacial infinity}).$$

(b) (i) Since the electromotive force must vanish everywhere,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{0} \implies \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \nabla\phi.$$

Utilizing the previous result, we have

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi = -\left[\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \nabla\phi\right] - \nabla\phi = -\times(\boldsymbol{\nabla} \times \mathbf{A}) = -\mathbf{v} \times \mathbf{B}.$$

Plugging this into Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, yields $\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B})$, as desired.

(ii) In analogy with the calculation in (a)(ii), we use the product rule to write $\frac{D}{Dt}(\mathbf{A} \cdot \mathbf{B}) = \left[\frac{D}{Dt}\mathbf{A}\right] \cdot \mathbf{B} + \mathbf{A} \cdot \left[\frac{D}{Dt}\mathbf{B}\right]$. Using the identities in the previous part, each of the convective derivatives can be written as

$$\begin{aligned}
\frac{D\mathbf{A}}{Dt} &= \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \\
&= \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \nabla\phi + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} && \text{(previous part)} \\
&= \nabla_{\mathbf{A}}(\mathbf{A} \cdot \mathbf{v}) - \nabla\phi && \text{(equation (5))}
\end{aligned}$$

and

$$\begin{aligned}
\frac{D\mathbf{B}}{Dt} &= \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} \\
&= \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} && \text{(previous part)} \\
&= \underbrace{\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{B})}_{\boldsymbol{\nabla}\cdot\mathbf{B}=0} - \underbrace{\mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{v})}_{=0,\ \text{incompressible}} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{v} && \text{(equation (6))}.
\end{aligned}$$

Using these one finds

$$\begin{aligned}
\frac{D}{Dt}\left[\mathbf{A} \cdot \mathbf{B}\right] &= \left[\nabla_{\mathbf{A}}(\mathbf{A} \cdot \mathbf{v}) - \nabla\phi\right] \cdot \mathbf{B} + \mathbf{A} \cdot \left[(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{v}\right] \\
&= \left[\nabla_{\mathbf{A}}(\mathbf{A} \cdot \mathbf{v}) - \nabla\phi\right] \cdot \mathbf{B} + \cdot \left[\nabla_{\mathbf{v}}(\mathbf{A} \cdot \mathbf{v})\right] \cdot \mathbf{B} \\
&= \mathbf{B} \cdot \left[\nabla(\mathbf{A} \cdot \mathbf{v}) - \nabla\phi\right] \\
&= \boldsymbol{\nabla} \cdot \left[\mathbf{B}(\mathbf{A} \cdot \mathbf{v} - \phi)\right] && \text{(using (3) and } \boldsymbol{\nabla} \cdot \mathbf{B} = 0\text{)},
\end{aligned}$$

as desired.

(iii) This is analogous to the calculation in the preceding part. Putting all of the pieces together one finds

$$\frac{d}{dt}W = \frac{d}{dt}\int_{\Omega} (\mathbf{A} \cdot \mathbf{B})\, dV$$

$$
\begin{aligned}
&= \int_\Omega \frac{D}{Dt}\left(\mathbf{A}\cdot\mathbf{B}\right)\, dV && \text{(by (a)(ii))} \\
&= \int_\Omega \boldsymbol{\nabla}\cdot\left\{\mathbf{B}\left(\mathbf{A}\cdot\mathbf{v}-\phi\right)\right\}\, dV && \text{(by (b)(ii))} \\
&= \int_{\partial\Omega} \left\{\mathbf{B}(\mathbf{A}\cdot\mathbf{v}-\phi\right\}\cdot d\mathbf{S} && \text{(Gauss's law).}
\end{aligned}
$$

If $\mathbf{B}$ vanishes at spatial infinity, then $\frac{d}{dt}W = 0$, which shows that W is a constant of motion.

4 Faraday's Law

(a) Following analogous steps to the procedure done in 3(a)(iii) (but here written explicitly in the language of differential forms), we pull back the time-varying region of integration to one that is fixed, $\Omega_0 \equiv \Omega(\tau = 0)$, via a diffeomorphism φ_t.

$$
\begin{aligned}
\frac{d}{d\tau}\int_{\Omega(\tau)} F &= \frac{d}{d\tau}\int_{\Omega(\tau)} \left(\frac{1}{2}F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu\right) \\
&= \frac{d}{d\tau}\int_{\varphi_\tau^{-1}(\Omega(\tau))} \varphi_\tau^* \left(\frac{1}{2}F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu\right) \\
&= \frac{d}{d\tau}\int_{\Omega_0} \frac{1}{2}F_{\mu\nu}(x(\xi))\frac{\partial x^\mu}{\partial \xi^\sigma}\frac{\partial x^\nu}{\partial \xi^\rho}\, d\xi^\sigma \wedge d\xi^\rho \\
&= \frac{1}{2}\int_{\Omega_0} \frac{d}{d\tau}\left(F_{\mu\nu}(x(\xi))\frac{\partial x^\mu}{\partial \xi^\sigma}\frac{\partial x^\nu}{\partial \xi^\rho}\right) d\xi^\sigma \wedge d\xi^\rho \\
&= \frac{1}{2}\int_{\Omega_0} \left[\left(\frac{d}{d\tau}F_{\mu\nu}(x(\xi))\right)\frac{\partial x^\mu}{\partial \xi^\sigma}\frac{\partial x^\nu}{\partial \xi^\rho}\right. \\
&\qquad \left. + F_{\mu\nu}(x(\xi))\left(\frac{d}{d\tau}\frac{\partial x^\mu}{\partial \xi^\sigma}\right)\frac{\partial x^\nu}{\partial \xi^\rho} + F_{\mu\nu}(x(\xi))\frac{\partial x^\mu}{\partial \xi^\sigma}\left(\frac{d}{d\tau}\frac{\partial x^\nu}{\partial \xi^\rho}\right)\right] d\xi^\sigma \wedge d\xi^\rho.
\end{aligned}
$$

Notice however that

$$
\frac{\partial}{\partial \tau}\frac{\partial x^\mu}{\partial \xi^\sigma} = \frac{\partial}{\partial \xi^\sigma}\underbrace{\left(\frac{\partial x^\mu}{\partial \tau}\right)}_{=V^\mu} = \frac{\partial x^\lambda}{\partial \xi^\sigma}\frac{\partial}{\partial x^\lambda}V^\mu,
$$

and analogously for the other terms. We can therefore write

$$
\begin{aligned}
\frac{d}{d\tau}\int_{\Omega(\tau)} F = \frac{1}{2}\int_{\Omega_0} &\left[V^\lambda \frac{\partial F_{\mu\nu}}{\partial x^\lambda}\left(\frac{\partial x^\mu}{\partial \xi^\sigma}\frac{\partial x^\nu}{\partial \xi^\rho}\right)\right. \\
&\left. + F_{\mu\nu}\frac{\partial V^\mu}{\partial x^\lambda}\left(\frac{\partial x^\lambda}{\partial \xi^\sigma}\frac{\partial x^\nu}{\partial \xi^\rho}\right) + F_{\mu\nu}\frac{\partial V^\nu}{\partial x^\lambda}\left(\frac{\partial x^\lambda}{\partial \xi^\rho}\frac{\partial x^\mu}{\partial \xi^\sigma}\right)\right] d\xi^\sigma \wedge d\xi^\rho \\
= \frac{1}{2}\int_{\Omega(\tau)} &\left[V^\lambda \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + F_{\lambda\nu}\frac{\partial V^\lambda}{\partial x^\mu} + F_{\mu\lambda}\frac{\partial V^\lambda}{\partial x^\nu}\right] \frac{\partial x^\mu}{\partial \xi^\sigma}\frac{\partial x^\nu}{\partial \xi^\rho}\, d\xi^\sigma \wedge d\xi^\rho
\end{aligned}
$$

$$= \frac{1}{2}\int_{\Omega_0} \underbrace{\left[V^\lambda \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + F_{\lambda\nu}\frac{\partial V^\lambda}{\partial x^\mu} + F_{\mu\lambda}\frac{\partial V^\lambda}{\partial x^\nu}\right]}_{=(\mathcal{L}_V F)_{\mu\nu}} dx^\mu \wedge dx^\nu.$$

In the second equality we have simply re-labeled indices; in the last, the integrand has been written back in terms of the original coordinates with a time-varying region of integration. This shows that $\frac{d}{d\tau}\int_{\Omega(\tau)} F = \int_{\Omega(\tau)} \mathcal{L}_V F$, as desired.[7]

(b) If τ is the proper time along the world-line of each element, then

$$\frac{dV^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dV^\mu}{dt} = \frac{1}{\sqrt{1-\mathbf{v}^2}}(1, \mathbf{v})$$

and

$$f = -\iota_V F = -\left(\frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu\right)(V^\sigma \frac{\partial}{\partial x^\sigma}, \cdot)$$
$$= -\frac{1}{2}F_{\mu\nu}\,(V^\sigma \delta^\mu{}_\sigma\, dx^\nu - V^\sigma \delta^\nu{}_\sigma\, dx^\mu) = F_{\mu\nu}V^\nu\, dx^\mu,$$

which is exactly the definition Lorentz-force 4-vector.

[7] What we have done here is essentially derive Leibniz's rule for 2-forms. Analogous results, which follow the same line of reasoning, can be derived for general p-forms. See Flanders, Harley "Differentiation Under the Integral Sign", for a proof of the general statement.