1 Infinitesimal Homotopy

The infinitesimal homotopy relation states

\[ \mathcal{L}_X \omega = (i_X + d_i X) \omega. \]  

(1)

Taking the exterior derivative of (1),

\[ d(\mathcal{L}_X \omega) = d(i_X + d_i X) \omega \] (using (1))
\[ = d^2 i_X \omega + d(i_X d \omega) \] (\( d^2 = 0 \))
\[ = (d i_X + i_X d)(d \omega) \] (add \( i_X d^2 \omega = 0 \))
\[ = \mathcal{L}_X (d \omega). \]

2 Magnetic Solid

(a) We need to verify that

\[ \dot{x} = \frac{\partial \epsilon(k)}{\partial k} - k \times \Omega \] \hfill (2a)
\[ \dot{k} = -\frac{\partial V}{\partial x} - \epsilon \dot{x} \times B \] \hfill (2b)

is indeed the Hamiltonian vector flow of \( H(x, k) = \epsilon(k) + V(x) \) with the symplectic form \( \omega \). This amounts to checking that

\[ dH = -i_{v_H} \omega = -\omega(v_H, \cdot) \] \hfill (3)

reproduces equations (2b) and (2b). First, expand \( dH \) in local coordinates we find

\[ dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial k^i} dk^i = \frac{\partial V(x)}{\partial x^i} dx^i + \frac{\partial \epsilon(k)}{\partial k^i} dk^i. \] \hfill (4)

Next, plugging in the velocity vector field,

\[ v_H = \dot{x}^i \frac{\partial}{\partial x^i} + \dot{k}^i \frac{\partial}{\partial k^i}, \] \hfill (5)
into \(-\omega(v_H, \cdot)\), we find\(^1\)

\[
-\omega(v_H, \cdot) = -\left\{ dk^i dx^i - \frac{e}{2} \epsilon_{ijk} B^i(x) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \Omega^i(k) dk^j dk^k \right\} \left( x^\ell \frac{\partial}{\partial x^\ell} + k^\ell \frac{\partial}{\partial k^\ell}, \cdot \right)
\]

\[
= -\dot{k} dx^i + \dot{x}_i dk^i + \frac{e}{2} \epsilon_{ijk} B^i(x) \left[ \dot{x}_j dx^k - \dot{x}_k dx^j \right] - \frac{1}{2} \epsilon_{ijk} \Omega^i(k) \left[ k_j dk^k - k_k dk^j \right]
\]

\[
= \left[ -\dot{k}_k - e \epsilon_{ijk} \dot{x}_i B^i(x) \right] dx^k + \left[ \dot{x}_k + \epsilon_{ijk} k_j \Omega^i(k) \right] dk^k,
\]

where in the last line I’ve relabeled indices, \(i \leftrightarrow j\), and used equation (4) to write the underset equalities. These equalities reproduce equations (2a) and (2b), as desired.

(b) We first check that \(\omega\) is closed.

\[
d\omega = d \left\{ dk^i dx^i - \frac{e}{2} \epsilon_{ijk} B^i(x) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \Omega^i(k) dk^j dk^k \right\}
\]

\[
= -\frac{e}{2} \epsilon_{ijk} \left( dB^i(x) \right) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \left( d\Omega^i(k) \right) dk^j dk^k
\]

\[
= -\frac{e}{2} \epsilon_{ijk} \left( \frac{\partial B^i(x)}{\partial x^\ell} dx^\ell \right) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \left( \frac{\Omega^i(k)}{\partial k^\ell} \right) dk^j dk^k
\]

\[
= -\frac{e}{2} \epsilon_{ijk} \left( \frac{\partial B^i(x)}{\partial x^\ell} \right) dx^i dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \left( \frac{\Omega^i(k)}{\partial k^\ell} \right) dk^i dk^j dk^k \quad \text{(antisymmetry} \implies \ell = i). \]

Now the product \(\epsilon_{ijk} dx^i dx^j dx^k\) is just proportional to \(dx^1 dx^2 dx^3\) (and likewise for the \(dk^i dk^j dk^k\)). Hence we can write

\[
d\omega \propto -\frac{e}{2} \left( \frac{\partial B^i(x)}{\partial x^\ell} \right) dx^1 dx^2 dx^3 + \frac{1}{2} \left( \frac{\Omega^i(k)}{\partial k^\ell} \right) dk^1 dk^2 dk^3.
\]

But this vanishes identically since \(\text{div}_X \mathbf{B} = \text{div}_k \Omega = 0\). Hence \(d\omega = 0\), as desired.

Now, we would like to use the definition of the Poisson bracket,\(^2\)

\[
\{H_1, H_2\} \overset{\text{def}}{=} \left. \frac{dH_2}{dt} \right|_{H_1} = \dot{H}_2,
\]

(6)

(or equivalently, \(\{H_1, H_2\} = -\dot{H}_1|_{H_2}\) since \(\{H_1, H_2\} = -\{H_2, H_1\}\)) to compute brackets consisting of \(x_i\) and \(k_j\) (and combinations thereof). However, the expressions are coupled in the sense that \(\dot{x}_i\) and \(\dot{k}_j\) themselves depend on time derivatives of the \(k_j\) and \(x_i\) respectively, whereas we want \(x_i = x_i(x, k)\) (and likewise for \(k_j\)).

\(^1\)For notational convenience, I will drop the wedge product and write \(dx^i dx^j\) in place of \(dx^i \wedge dx^j\). I will also simply write \(dx^i\) rather than \(dx^i(\cdot)\), although it is still implied that the dual basis elements act on basis elements (of the tangent space).

\(^2\)As noted in the text, there are other definitions which differ by a minus sign, so be careful (and consistent).
We can however massage equations (2a) and (2b) to get the desired expressions by noting the following dot product equalities,

\[ \begin{align*}
(2a) & \implies \mathbf{x} \cdot \Omega = \frac{\partial \epsilon(k)}{\partial k} \cdot \Omega - (\dot{k} \times \Omega) \cdot \Omega = 0 \\
(2b) & \implies \dot{k} \cdot \mathbf{B} = -\frac{\partial V(x)}{\partial x} \cdot \mathbf{B} - (e\dot{x} \times \mathbf{B}) \cdot \mathbf{B} = 0
\end{align*} \]  
(7a)  
(7b)

Plugging (2b) into (2a),

\[ \begin{align*}
\dot{x} &= \frac{\partial \epsilon(k)}{\partial k} - \left[ \frac{\partial V(x)}{\partial x} - e\dot{x} \times \mathbf{B} \right] \times \Omega \\
&= \frac{\partial \epsilon(k)}{\partial k} + \frac{\partial V(x)}{\partial x} \times \Omega - \left[ e\dot{x}(\Omega \cdot \mathbf{B}) - \mathbf{B}(e\dot{x} \cdot \Omega) \right] \quad \text{(BAC-CAB Rule)} \\
&= \frac{\partial \epsilon(k)}{\partial k} + \frac{\partial V(x)}{\partial x} \times \Omega - \left[ e\dot{x}(\Omega \cdot \mathbf{B}) - \mathbf{B}(e\frac{\partial \epsilon(k)}{\partial k} \cdot \Omega) \right] \quad \text{(equation (7a)).}
\end{align*} \]

This rearranges to

\[ \dot{x}(1 + e\mathbf{B} \cdot \Omega) = \frac{\partial \epsilon(k)}{\partial k} + \frac{\partial V(x)}{\partial x} \times \Omega + \mathbf{B} \left( e\frac{\partial \epsilon(k)}{\partial k} \cdot \Omega \right). \]  
(8)

The analogous procedure for \( \dot{k} \) yields

\[ \dot{k}(1 + e\mathbf{B} \cdot \Omega) = -\frac{\partial V(x)}{\partial x} - e\frac{\partial \epsilon(k)}{\partial k} \times \mathbf{B} - \mathbf{B} \left( e\frac{\partial V(x)}{\partial x} \cdot \mathbf{B} \right). \]  
(9)

This leaves us with expressions for \( x_i \) and \( k_j \) which are solely in terms of the \( x \) and \( k \). We can then take \( H_1 \) and \( H_2 \) to be the desired functions (which specifies the form of \( \epsilon(k) \) and \( V(x) \)) and simply evaluate them using definition (6).

Letting \( H_1 = x_i \) and \( H_2 = x_j \) in the definition, equation (8) yields

\[ \{x_i, x_j\} = -\dot{x}_i = -\frac{\epsilon_{ijk} \Omega_k}{(1 + e\mathbf{B} \cdot \Omega)}. \]

The remaining two Poisson brackets follow by the same procedure. Summarizing, one finds

\[ \begin{align*}
\{x_i, x_j\} &= -\frac{\epsilon_{ijk} \Omega_k}{(1 + e\mathbf{B} \cdot \Omega)}, \\
\{x_i, k_j\} &= -\frac{\delta_{ij} + eB_i \Omega_j}{(1 + e\mathbf{B} \cdot \Omega)}, \\
\{k_i, k_j\} &= \frac{\epsilon_{ijk} e B_k}{(1 + e\mathbf{B} \cdot \Omega)}.
\end{align*} \]

(c) The conserved phase-space volume \( \omega^3/3! \) can be computed by direct calculation. Note that terms like \( dx^i \land dx^j \land dx^k \land dx^l \) vanish since necessarily there will be one repeated index for
three spatial dimensions (an analogous argument holds for the k’s). Hence

\[ \omega^3 = dx^i dx^j dx^k dk^i dk^j dk^k + 3! (dk^i dx^i) \left( -\frac{e}{2} \epsilon_{i' j' k'} B^{i' j'} dx^{i'} dx^{j'} \right) \left( \frac{1}{2} \epsilon_{i'' j'' k''} \Omega^{i''} dk^{i''} dk^{j''} \right) \]

\[ = dk^i dk^j dk^k dx^i dx^j dk^k - 3! \frac{e}{4} \left( \epsilon_{i' j' k'} \epsilon_{i'' j'' k''} \Omega^{i''} \right) dk^i dx^i dx^j dk^j dk^k \]

\[ = 3! \left[ 1 + (eB \cdot \Omega) \right] d^3k d^3x, \]

which implies \( \omega^3/3! = (1 + eB \cdot \Omega) d^3k d^3x, \) as desired.

### 3 Non-abelian Gauge Fields as Matrix-valued Forms

(i) Given \( A = A_\mu dx^\mu, \) write

\[ 2A^2 = A_\mu A_\nu dx^\mu dx^\nu + A_\mu A_\nu dx^\mu dx^\nu = (A_\mu A_\nu - A_\nu A_\mu) dx^\mu dx^\nu \]

\[ \implies A^2 = \frac{1}{2} [A_\mu, A_\nu] dx^\mu dx^\nu, \]

where in the last equality I’ve used \( dx^\mu dx^\nu = -dx^\nu dx^\mu \) and relabeled indices. Similarly, one finds \( dA = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu dx^\nu, \) so that

\[ F = dA + A^2 = \left( \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \right) dx^\mu dx^\nu. \]

(ii) Using the definition of the gauge-covariant derivatives,

\[ \nabla_\mu = \partial_\mu - A_\mu, \]

one finds

\[ [\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \]

\[ = (\partial_\mu - A_\mu)(\partial_\nu - A_\nu) - (\partial_\nu - A_\nu)(\partial_\mu - A_\mu) \]

\[ = \partial_\mu \partial_\nu + (\partial_\mu A_\nu) + A_\nu \partial_\mu + A_\mu \partial_\nu + A_\mu A_\nu - \partial_\nu \partial_\mu - (\partial_\nu A_\mu) - A_\mu \partial_\nu - A_\nu \partial_\mu - A_\nu A_\mu \]

\[ = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + A_\mu A_\nu - A_\nu A_\mu \]

\[ = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \]

\[ = F_{\mu\nu} \quad \text{(by part (i)).} \]

(iii) Let \( g \) be an invertable matrix and \( \delta g \) be a matrix describing a small change in \( g \) (we assume
\(g + \delta g\) is still invertable).

\[
(g + \delta g)(g^{-1} + \delta(g^{-1})) = \text{id} \implies \frac{gg^{-1} + g(\delta g)^{-1} + (\delta g)g^{-1} + (\delta g)\delta(g^{-1})}{\text{id}} = \text{id} \\
\implies (\delta g)^{-1} = -g^{-1}(\delta g)g^{-1}.
\]

Alternatively, by demanding there is no variation in the identity, we find \(0 = \delta(gg^{-1}) = (\delta g)g^{-1} + g\delta(g^{-1}) \implies \delta(g^{-1}) = (\delta g)^{-1} = -g^{-1}(\delta g)g^{-1}.

(iv) Suppose that the matrix-valued gauge field is a “pure gauge”; i.e., that \(A = g^{-1} dg\). Then

\[
dA = d(g^{-1} dg) = -g^{-1} dg g^{-1} dg = (g^{-1} dg)^{-1}.
\]

This shows that

\[
F = dA + A^2 = -\left( g^{-1} dg \right)^2 + \left( g^{-1} dg \right)^2 = 0,
\]

as desired.

(v) Under a gauge transformation,

\[
A_{\mu} \mapsto A_{\mu}^g = g^{-1} A_{\mu} g + g^{-1}(\partial_{\mu} g).
\]

Therefore, the covariant derivative transforms like

\[
\nabla_{\mu} \mapsto \nabla_{\mu}^g = \partial_{\mu} + A_{\mu}^g = g^{-1} g \partial_{\mu} + g^{-1}(\partial_{\mu} g) + g^{-1} A_{\mu} = g^{-1}(\partial_{\mu} + A_{\mu}) g.
\]

In the last equality, we have used the fact that the derivative acts to the right along with the chain rule (in reverse). Hence, \(\nabla_{\mu} \mapsto g^{-1} \nabla_{\mu} g\) under a gauge transformation. Using the result from part (ii) \((F_{\mu \nu} = [\nabla_{\mu}, \nabla_{\nu}]\)), we can easily find how \(F_{\mu \nu}\) behaves when transformed.

\[
F_{\mu \nu} = [\nabla_{\mu}, \nabla_{\nu}] \mapsto [g^{-1} \nabla_{\mu} g, g^{-1} \nabla_{\nu} g] = g^{-1} [\nabla_{\mu}, \nabla_{\nu}] g = g^{-1} F_{\mu \nu} g,
\]

as desired.

(vi) To show the Bianchi identity, we simply take the exterior derivative of \(F\).

\[
dF = d(dA + A^2) \\
= d^2A + (dA)A - A(dA) \\
= (F - A^2)A - A(F - A^2) \\
= FA - AF.
\]

This rearranges to \(dF - FA + AF = 0\), as desired.
(vii) Next, use the Bianchi identity to show that the 4-form is closed.

\[ d \tr(F^2) = \tr(dF^2) \]
\[ = \tr((dF)F + F(dF)) \]
\[ = \tr((FA - AF)F + F(FA - AF)) \quad \text{(Bianchi identity)} \]
\[ = \tr(EAFF - AFF + FFA - EAF) \]
\[ = - \tr(AFF) + \tr(AFF) \quad \text{(cyclic perm.)} \]
\[ = 0. \]

Note that in this case the cyclic permutation of matrix-valued forms is also even so that this operation doesn’t change the sign in the wedge product.\(^3\)

\[ \tr(FFA) = \tr(F_{\mu\nu}F_{\gamma\delta}A_\lambda)dx^\mu dx^\nu dx^\gamma dx^\delta dx^\lambda \]
\[ = (-1)^4 \tr(A_\lambda F_{\mu\nu}F_{\gamma\delta})dx^\lambda dx^\mu dx^\nu dx^\gamma dx^\delta \]
\[ = \tr(AFF). \]

(viii) Before showing that

\[ \tr(F^2) = d \left\{ \tr \left( AdA + \frac{2}{3} A^3 \right) \right\}, \quad (10) \]

first note that \( \tr(A^4) = 0 \) since

\[ \tr(A^4) = \tr(A_\mu A_\nu A_\gamma A_\delta)dx^\mu dx^\nu dx^\gamma dx^\delta \]
\[ = (-1)^3 \tr(A_\delta A_\mu A_\nu A_\gamma)dx^\delta dx^\mu dx^\nu dx^\gamma \]
\[ = - \tr(A^4). \]

In the second equality, we have cyclically permuted matrices under the trace; however, unlike the product of three matrices in part (vii), this is obtained via an odd permutation which introduces a minus in the 4-form. With this, we can show (10) beginning from the right-hand-side.

\[ d \left\{ \tr \left( AdA + \frac{2}{3} A^3 \right) \right\} = \tr \left\{ (dA)^2 + \frac{2}{3} [(dA)^2 + (dA)A + A^2(dA)] \right\} \]
\[ = \tr \left\{ (dA)^2 + A^2(dA) + A(dA)A + A^2(dA) \right\} \quad \text{(tr(A^4) = 0)} \]
\[ = \tr \left\{ (dA + A^2)^2 \right\} \]
\[ = \tr \left\{ F^2 \right\}. \]

Hence \( \tr(F^2) \) is also exact. In the first line, I’ve used the cyclic property of the trace to write \( A(dA)A \) in a symmetric way.

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\(^3\)Again, the wedge product is implicit.