Physics 509 Homework 1

Spring 2023

Professor M. Stone
University of Illinois

1 Counting Indices

(i) The dimension of the space of skew-symmetric covariant tensors with p indices is equal to the number of different ways we can choose p elements from the d elements spanning the space; i.e.,

$$\binom{d}{p} = \frac{d!}{p!(d-p)!}.$$

(ii) In the symmetric case, we again have d choose p basis elements, but *with replacements*. This is usually denoted by

$$\left(\!\!\binom{d}{p}\!\!\right) = \binom{d+p-1}{p} = \frac{(d+p-1)!}{p!(d-1)!}.$$

Without knowing the formula for subset selection with repetitions, one could derive this result as follows. Since we are in the symmetric case, repeated indices are allowed, but any permutation of the same indices results in the same basis element. We can however sort indices in non-decreasing order to determine the total number of basis elements. This procedure shows that the dimension of the space of symmetric covariant tensors with p indices is just the number non-decreasing sequences of length p using d integers. The number of non-decreasing sequences (allowing repetitions) of length p among d integers can be found by a standard "bars and stars" argument: among $p+d$ slots, place p bars. The number of stars to the left of each bar gives the corresponding index. Since indices range from 1 to d, the first slot must be occupied by a star. Of the remaining $d+p-1$ slots, we are left to choose p of them for the bars. Hence the dimension of the space is given by

$$\binom{d+p-1}{p} = \frac{(d+p-1)!}{p!(d-1)!}.$$

2 Quantum Entanglement

(i) Suppose $\mathbf{a}$ is a nonzero decomposable tensor. Then for some p and q, we have $a^{pq} = x^p x^q \neq 0$, with the remaining coefficients satisfying

$$a^{ij} = x^i y^j = \frac{x^i y^j x^p y^q}{x^p y^q} = \frac{a^{pj} a^{iq}}{a^{pq}}. \tag{1}$$

However, not all of these are constraints since

$$a^{iq} = \frac{\cancel{a^{pq}}a^{iq}}{\cancel{a^{pq}}} = a^{iq} \qquad \text{and} \qquad a^{pj} = \frac{\cancel{a^{pj}}a^{pq}}{\cancel{a^{pq}}} = a^{pj};$$

i.e., $i = p$ or $j = q$ do not result in constraints. The total number of constraints is then $(n-1)(m-1)$.

A more physical argument is to note that $\mathbf{a} = a^{ij}\mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)} \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ has mn components whereas $\mathbf{a} = \mathbf{x} \otimes \mathbf{y} = x^i y^j \mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)}$ has $m+n$ components. Therefore, the number of constraints a decomposable state must satisfy is $mn - (m+n) + 1$, where one constraint has been added for the normalization of states.

(ii) Rewriting equation (1) yields

$$a^{ij}a^{pq} = a^{pj}a^{iq} \implies \det\begin{pmatrix} a^{ij} & a^{iq} \\ a^{pj} & a^{pq} \end{pmatrix} = a^{ij}a^{pq} - a^{pj}a^{iq} = 0,$$

where the last equality follows since our state is decomposable and satisfies (1). Note that this yields a total of m^2n^2 equations since i and p range from $1, 2\ldots, m$ and j and q from $1, 2, \ldots, n$.

(iii) If $\mathbf{a}$ is the zero tensor, then it is trivially decomposable. For any nonzero tensor, it suffices to pick a single nonzero a^{pq} and choose only the constraints with $i = 1, 2, \ldots, m$ $(i \neq p)$ and $j = 1, 2, \ldots, n$ $(j \neq q)$. This yields the same $(m-1)(n-1)$ constraints as those in equation (1).

It remains to show that

$$a^{ij} = \frac{a^{pj}a^{iq}}{a^{pq}} \iff a^{ij} = x^i y^j,$$

for all i and j. The necessary condition ($\Longleftarrow$) was shown in part (i). The sufficient condition ($\implies$) can be shown by first choosing x^p and y^q such that $x^p y^q = a^{pq}$. The remaining conditions, $a^{ij} = x^i y^j$, are then satisfied by defining

$$x^i = \frac{a^{iq}}{a^{pq}}x^p \qquad \text{and} \qquad y^j = \frac{a^{pj}}{a^{pq}}y^q,$$

so that

$$a^{ij} = \frac{a^{pj}a^{iq}}{a^{pq}} = \frac{\left(\frac{y^j a^{pq}}{y^q}\right)\left(\frac{x^i a^{pq}}{x^p}\right)}{a^{pq}} = \frac{x^i y^j a^{pq}}{\underbrace{x^p y^q}_{=a^{pq}}} = x^i y^j.$$

3 Symmetric Integration

We first show that

$$I_{\alpha\beta\gamma\delta} = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} (k_\alpha k_\beta k_\gamma k_\delta) f(k^2) \tag{2}$$

is invariant under (proper) orthogonal transformations. For a general $\mathcal{O} \in SO(n)$,

$$\begin{aligned}
&\mathcal{O}_{\alpha'\alpha}\mathcal{O}_{\beta'\beta}\mathcal{O}_{\gamma'\gamma}\mathcal{O}_{\delta'\delta} I_{\alpha\beta\gamma\delta} \\
&\qquad = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} (\mathcal{O}_{\alpha'\alpha}k_\alpha)(\mathcal{O}_{\beta'\beta}k_\beta)(\mathcal{O}_{\gamma'\gamma}k_\gamma)(\mathcal{O}_{\delta'\delta}k_\delta) f(k^2) \\
&\qquad = \int_{\mathcal{O}^{-1}(\mathbb{R}^n)} \left|\det \mathcal{O}^{-1}\right| \frac{d^n q}{(2\pi)^n} q_{\alpha'} q_{\beta'} q_{\gamma'} q_{\delta'} f(q^2) && (\mathbf{q} \equiv \mathcal{O}\mathbf{k}) \\
&\qquad = \int_{\mathbb{R}^n} \frac{d^n q}{(2\pi)^n} q_{\alpha'} q_{\beta'} q_{\gamma'} q_{\delta'} f(q^2) && (\det \mathcal{O}^{-1} = 1),
\end{aligned}$$

which is numerically equal to $I_{\alpha\beta\gamma\delta}$ up to relabelling. Note, we have used the fact that k^2 is left invariant under an orthogonal transformation since

$$q^2 = \mathbf{q}^t\mathbf{q} = (\mathcal{O}\mathbf{k})^t(\mathcal{O}\mathbf{k}) = \mathbf{k}^t \underbrace{\mathcal{O}^t\mathcal{O}}_{=\mathrm{id}} \mathbf{k} = k^2.$$

Because of invariance (see (10.87) in the textbook), we can write

$$I_{\alpha\beta\gamma\delta} = a\delta_{\alpha\beta}\delta_{\gamma\delta} + b\delta_{\alpha\gamma}\delta_{\delta\beta} + c\delta_{\alpha\delta}\delta_{\beta\gamma} + d\epsilon_{\alpha\beta\gamma\delta}$$

for some coefficients a, b, c, and d. Since $I_{\alpha\beta\gamma\delta}$ is completely symmetric in all its indices, $a = b = c$. Additionally, since $\epsilon_{\alpha\beta\gamma\delta}$ is completely asymmetric, $d = 0$. We can then write

$$I_{\alpha\beta\gamma\delta} = A(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\delta\beta} + \delta_{\alpha\delta}\delta_{\beta\gamma}).$$

A can be determined by (note the implicit summations over repeated indices)

$$\begin{aligned}
I_{\alpha\alpha\gamma\gamma} &= I_{\alpha\beta\gamma\delta}\delta_{\alpha\beta}\delta_{\gamma\delta} \\
&= A(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\delta\beta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\delta_{\alpha\beta}\delta_{\gamma\delta} \\
&= A(\underbrace{\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\alpha\beta}\delta_{\gamma\delta}}_{=n^2} + \underbrace{\delta_{\alpha\gamma}\delta_{\delta\beta}\delta_{\alpha\beta}\delta_{\gamma\delta}}_{=n} + \underbrace{\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{\alpha\beta}\delta_{\gamma\delta}}_{=n}) \\
&= A(n^2 + 2n).
\end{aligned}$$

Therefore,

$$A = \frac{1}{n(n+2)} \int \frac{d^n k}{(2\pi)^2} (k^2)^2 f(k^2).$$

For the analogous rank-five tensor,

$$I_{\alpha\beta\gamma\delta\epsilon} = \int \frac{d^n k}{(2\pi)^2} \underbrace{(k_\alpha k_\beta k_\gamma k_\delta k_\epsilon)}_{\text{odd}} \underbrace{f(k^2)}_{\text{even}} = 0.$$

The integral clearly vanishes since we are integrating an odd function over symmetric bounds.

4 Leonardo da Vinci's Problem II

First, note that along the centroid ($x = y = 0$) $\eta_y < 0$ for $z > 0$, so the deformation is indeed downward. To show that the R in the problem does in fact coincide with the radius of curvature (of the line of centroids), note that in the vicinity of any point on the line of centroids, we can write $y_c(z) = -\frac{z^2}{2R}$. Evaluating the derivatives at $z = 0$, one finds (using the definition of the radius of curvature)

$$\text{Radius of curvature} \equiv \left| \frac{(1 + y_c')^{3/2}}{y_c''} \right| = \left| \frac{1}{-1/R} \right| = R.$$

The strain tensor e_{ij} is defined as

$$e_{ij} = \frac{1}{2} \left(\frac{\partial \eta_j}{\partial x_i} + \frac{\partial \eta_i}{\partial x_j} \right). \tag{3}$$

Using the deformations given in the problem,

$$\begin{aligned} e_{xx} &= -\frac{\sigma}{R} y & e_{yy} &= -\frac{\sigma}{R} y & e_{zz} &= \frac{y}{R} \\ e_{xy} &= 0 & e_{xz} &= 0 & e_{yz} &= 0. \end{aligned}$$

The remaining elements are determined by symmetry ($e_{ij} = e_{ji}$). From equation (10.103) in the text, we find the stress tensor elements are

$$\sigma_{zz} = Y \left(\frac{\eta_z}{z} \right) = \frac{Y}{R} y,$$

while $\sigma_{xx} = \sigma_{yy} = 0$ since there are no forces acting on the sides of the beam. These statements can be explicitly verified from the general definition of the stress tensor,

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}, \tag{4}$$

where λ and μ are the Lamé constants. Plugging in the strain tensor elements to (4) and using the relationship $\lambda(1 - 2\sigma) + 2\mu = 2\mu(1 + \sigma)$ (see equations (10.108) and (10.112) in the textbook), one immediately calculates the stress tensor elements.

Finally, we calculate the total elastic energy.

$$E = \frac{1}{2}\iiint_{\text{beam}} e_{ij}\,\underbrace{c_{ijkl}e_{kl}}_{=\sigma_{ij}}\,d^3x = \frac{1}{2}\int_\Gamma\int_L e_{zz}\sigma_{zz}\,ds dx dy$$

$$= \underbrace{\left(\int_\Gamma y^2\,dxdy\right)}_{=I}\left(\int_L \frac{Y}{2}\left(\frac{1}{R}\right)^2\,ds\right) \approx \int \frac{YI}{2}(y'')^2\,dz,$$

where in the third equality I've plugged in our earlier result $\sigma_{zz} = \frac{Y}{R}y$.

5 Maxwell Stress

The Maxwell stress tensor is given by

$$\Pi_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}\left|\mathbf{E}\right|^2\right) + \mu_0\left(H_iH_j - \frac{1}{2}\delta_{ij}\left|\mathbf{H}\right|^2\right). \tag{5}$$

Maxwell's equations in component form (and written in terms of $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$) become

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad\Longleftrightarrow\quad \partial_iE_i = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's Law)} \tag{6a}$$

$$\nabla\cdot\mathbf{H} = 0 \quad\Longleftrightarrow\quad \partial_iH_i = 0 \tag{6b}$$

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \quad\Longleftrightarrow\quad \epsilon_{ijk}\partial_iE_j = -\mu_0\partial_tH_k \qquad \text{(Faraday's Law)} \tag{6c}$$

$$\nabla\times\mathbf{H} = \mathbf{j} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \quad\Longleftrightarrow\quad \epsilon_{ijk}\partial_jH_k = j_k + \epsilon_0\partial_tE_k \qquad \text{(Ampère's Law).} \tag{6d}$$

To show the identity

$$\partial_j\Pi_{ij} = \underbrace{(\rho\mathbf{E} + \mathbf{j}\times\mathbf{B})_i}_{\equiv\text{(I)}} + \underbrace{\frac{\partial}{\partial t}\left\{\frac{1}{c^2}\left(\mathbf{E}\times\mathbf{H}\right)_i\right\}}_{\equiv\text{(II)}}, \tag{7}$$

it is easiest to expand the RHS. Using index notation,

$$\text{(I)} = \rho E_i + \epsilon_{ijk}j_jB_k,$$

and (using the shorthand $\partial_t = \frac{\partial}{\partial t}$)

$$
\begin{aligned}
(\mathrm{II}) &= \frac{1}{c^2}\epsilon_{ijk}\left(E_j\partial_t H_k + H_k\partial_t E_j\right) \\
&= \frac{1}{c^2}\epsilon_{ijk}\left[E_j\left(-\frac{1}{\mu_0}\epsilon_{ki'j'}\partial_{i'}E_{j'}\right) + \frac{1}{\epsilon_0}H_k\left(-j_j + \epsilon_{ji'k'}\partial_{i'}H_{k'}\right)\right] && \text{((6c) and (6d))} \\
&= -\epsilon_{ijk}j_jB_k - \epsilon_0\underbrace{\epsilon_{ijk}\epsilon_{i'j'k}}_{=(\delta_{ii'}\delta_{jj'}-\delta_{ij'}\delta_{ji'})}E_j\partial_{i'}E_{j'} - \mu_0\underbrace{\epsilon_{ijk}\epsilon_{i'jk'}}_{=(\delta_{ii'}\delta_{kk'}-\delta_{ik'}\delta_{ki'})}H_k\partial_{i'}H_{k'} && \text{(homework 0, 2(c))} \\
&= -\epsilon_{ijk}j_jB_k - \epsilon_0(E_j\partial_iE_j - E_j\partial_jE_i) - \mu_0(H_k\partial_iH_k - H_k\partial_kH_i).
\end{aligned}
$$

Combining (I) + (II) and relabelling indices,

$$
\begin{aligned}
\partial_j\Pi_{ij} &= \rho E_i - \epsilon_0(E_j\partial_iE_j - E_j\partial_jE_i) - \mu_0(H_k\partial_iH_k - H_k\partial_kH_i) \\
&= \epsilon_0(\partial_jE_j)E_i + \mu_0(\partial_jH_j)H_i - \epsilon_0(E_k\partial_iE_k - E_j\partial_jE_i) && \text{((6a) and (6b))} \\
&\qquad - \mu_0(H_k\partial_iH_k - H_j\partial_jH_i) \\
&= \epsilon_0\big[(\partial_jE_j)E_i + E_j\partial_jE_i - E_k\partial_iE_k\big] + \mu_0\big[(\partial_jH_j)H_i + H_j\partial_jH_i - H_k\partial_iH_k\big] \\
&= \partial_j\left\{\epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}\left|\mathbf{E}\right|^2\right) + \mu_0\left(H_iH_j - \frac{1}{2}\delta_{ij}\left|\mathbf{H}\right|^2\right)\right\},
\end{aligned}
$$

as desired.