1) **Buckyball spectrum.**: Consider the symmetry group of the C\textsubscript{60} buckyball molecule illustrated on page 194 of the notes.

a) Starting from the character table of the orientation-preserving icosohedral group \( Y \) (table 5.3), and using the fact that the \( \mathbb{Z}_2 \) parity inversion \( \sigma : \mathbf{r} \rightarrow -\mathbf{r} \) combines with \( g \in Y \) so that \( D^J_g(\sigma g) = D^J_g(g) \), whilst \( D^J_g(u\sigma g) = -D^J_g(u\sigma g) \), write down the character table of the extended group \( Y_h = Y \times \mathbb{Z}_2 \) that acts as a symmetry on the C\textsubscript{60} molecule. There are now ten conjugacy classes, and the ten representations will be labelled \( A_g, A_u, \ldots \). Verify that your character table has the expected row-orthogonality properties.

b) By counting the number of atoms left fixed by each group operation, compute the compound character of the action of \( Y_h \) on the C\textsubscript{60} molecule. (Hint: Examine the pattern of panels on a regulation soccer ball, and deduce that four carbon atoms are left unmoved by operations in the class \( \sigma C_2 \).)

c) Use your compound character from part b), to show that the 60-dimensional Hilbert space decomposes as

\[
\mathcal{H}_{C_{60}} = A_g \oplus T_{1g} \oplus 2T_{1u} \oplus T_{2g} \oplus 2T_{2u} \oplus 2G_g \oplus 2G_u \oplus 3H_g \oplus 2H_u,
\]

consistent with the energy-levels sketched in figure 5.3.

2) **Matrix commutators**:

a) Let \( \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \) be hermitian matrices. Show that if we define \( \hat{\lambda}_3 \) by the relation \( [\hat{\lambda}_1, \hat{\lambda}_2] = i\hat{\lambda}_3 \), then \( \hat{\lambda}_3 \) is also a hermitian matrix.

b) For the Lie group O(\( n \)), the matrices “\( i\hat{\lambda} \)” are real \( n \)-by-\( n \) skew symmetric matrices. Show that if \( A_1 \) and \( A_2 \) are real skew symmetric matrices, then so is \( [A_1, A_2] \).

c) For the Lie group Sp(2\( n \), \( \mathbb{R} \)), the \( i\hat{\lambda} \) matrices are of the form

\[
A = \begin{pmatrix}
a & b \\
c & -a^T
\end{pmatrix}
\]

where \( a \) is a real \( n \)-by-\( n \) matrix and \( b \) and \( c \) are symmetric (\( a^T = a \) and \( b^T = b \)) real \( n \)-by-\( n \) matrices. Show that the commutator of any two matrices of this form is also of this form.

3 **Euler angles and SU(2)**: Parametrize the elements of SU(2) as

\[
U = \exp\{-i\phi \hat{\sigma}_3/2\} \exp\{-i\theta \hat{\sigma}_2/2\} \exp\{-i\psi \hat{\sigma}_3/2\} =
\begin{pmatrix}
e^{-i\phi/2} & 0 \\
0 & e^{i\phi/2}
\end{pmatrix}
\begin{pmatrix}
\cos \theta/2 & -\sin \theta/2 \\
\sin \theta/2 & \cos \theta/2
\end{pmatrix}
\begin{pmatrix}
e^{-i\psi/2} & 0 \\
0 & e^{i\psi/2}
\end{pmatrix};
\]

\[
U =
\begin{pmatrix}
e^{-i(\phi+\psi)/2} \cos \theta/2 & -e^{i(\psi-\phi)/2} \sin \theta/2 \\
e^{i(\phi-\psi)/2} \sin \theta/2 & e^{i(\psi+\phi)/2} \cos \theta/2
\end{pmatrix}.
\]
a) Show that Hopf : $S^3 \rightarrow S^2$ is the projection of $S^3 \simeq SU(2)$ onto the coset space $S^2 \simeq SU(2)/U(1)$, where $U(1)$ is the subgroup $\{e^{i\psi}\hat{\sigma}_3/2\}$. Conclude that Hopf takes $(\theta, \phi, \psi) \rightarrow (\theta, \phi)$, where $\theta$ and $\phi$ are spherical polar co-ordinates on the two-sphere.

b) Show that $U^{-1}dU = -\frac{i}{2} \hat{\sigma}_i \Omega^i_L$, where

\[
\begin{align*}
\Omega^1_L &= \sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi, \\
\Omega^2_L &= \cos \psi \, d\theta + \sin \theta \sin \psi \, d\phi, \\
\Omega^3_L &= d\psi + \cos \theta \, d\phi.
\end{align*}
\]

Compare these 1-forms with the components

\[
\begin{align*}
\omega_x &= \sin \psi \, \dot{\theta} - \sin \theta \cos \psi \, \dot{\phi}, \\
\omega_y &= \cos \psi \, \dot{\theta} - \sin \theta \sin \psi \, \dot{\phi}, \\
\omega_z &= \dot{\psi} + \cos \theta \, \dot{\phi}.
\end{align*}
\]

of the angular velocity $\omega$ of a body with respect to the body-fixed $XYZ$.

c) (Optional) Now show that $dUU^{-1} = -\frac{i}{2} \hat{\sigma}_i \Omega^i_R$, where

\[
\begin{align*}
\Omega^1_R &= -\sin \phi \, d\theta + \sin \theta \cos \psi \, d\psi, \\
\Omega^2_R &= \cos \phi \, d\theta + \sin \theta \sin \psi \, d\psi, \\
\Omega^3_R &= d\phi + \cos \theta \, d\psi,
\end{align*}
\]

Compare these 1-forms with components $\omega_x, \omega_y, \omega_z$ of the same angular velocity vector $\omega$, but now with respect to the space-fixed $xyz$ frame.

4) Class and group volume:

a) In the lecture notes I claimed that the volume fraction of the group $SU(2)$ occupied by rotations through angles lying between $\theta$ and $\theta + d\theta$ is $\sin^2(\theta/2)d\theta/\pi$. By considering the geometry of the three-sphere, show that this is correct.

b) Show that

\[
\int_{SU(2)} \text{tr}[(U^{-1}dU)^3] = 24\pi^2.
\]
c) Suppose we have a map \( g : \mathbb{R}^3 \to \text{SU}(2) \) such that \( g(x) \) goes to the identity element at infinity. Consider the integral 
\[
S[g] = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \text{tr} \left[ (g^{-1}dg)^3 \right],
\]
where the 3-form \( \text{tr} (g^{-1}dg)^3 \) is the pull-back to \( \mathbb{R}^3 \) of the form \( \text{tr} [(U^{-1}dU)^3] \) on SU(2). Show that if we vary \( g \to g + \delta g \), then 
\[
\delta S[g] = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} d \left\{ 3 \text{tr} \left[ (g^{-1}\delta g)(g^{-1}dg)^2 \right] \right\} = 0,
\]
and so \( S[g] \) is topological invariant of the map \( g \). Conclude that the functional \( S[g] \) is an integer, that integer being the Brouwer degree, or winding number, of the map \( g : S^3 \to S^3 \).

5) **Campbell-Baker-Hausdorff Formulae**: Here are some useful formula for working with exponentials of matrices that do not commute with each other.

a) Let \( X \) and \( Y \) be matrices. Show that
\[
e^{tX}Y e^{-tX} = Y + t[X,Y] + \frac{1}{2}t^2[X,[X,Y]] + \cdots,
\]
the terms on the right being the series expansion of \( \exp[\text{ad}(tX)]Y \). A proof is sketched in a footnote in the lecture notes, but I want you to fill in the details.

b) Let \( X \) and \( \delta X \) be matrices. Show that
\[
e^{-X} e^{X+\delta X} = 1 + \int_0^1 e^{-tx} \delta X e^{tx} dt + O \left[ (\delta X)^2 \right] = 1 + \delta X - \frac{1}{2}[X,\delta X] + \frac{1}{3!}[X,[X,\delta X]] + \cdots
\]
\[
= 1 + \left( 1 - \frac{e^{-\text{ad}(X)}}{\text{ad}(X)} \right) \delta X + O \left[ (\delta X)^2 \right]
\]

c) By expanding out the exponentials, show that
\[
e^X Y e^Y = e^{X+Y} + \frac{1}{2}[X,Y] + \text{higher},
\]
where “higher” means terms higher order in \( X, Y \). The next two terms are, in fact, \( \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] \).

6) **SU(3)**: Here are some simple results that come from playing with the Gell-Mann lambda matrices, as well as practice at decomposing tensor products.

The totally antisymmetric structure constants, \( f_{ijk} \), and a set of totally symmetric constants \( d_{ijk} \) are defined by
\[
f_{ijk} = \frac{1}{2} \text{tr} \left( \lambda_i [\lambda_j, \lambda_k] \right), \quad d_{ijk} = \frac{1}{2} \text{tr} \left( \lambda_i \{\lambda_j, \lambda_k\} \right).
\]
Let \( D_{ij}^8(g) \) be the matrices representing \( SU(3) \) in “8” — the eight-dimensional adjoint representation.

a) Show that

\[
\begin{align*}
f_{ijk} &= D_{il}^8(g)D_{jm}^8(g)D_{kn}^8(g)f_{lmn}, \\
d_{ijk} &= D_{il}^8(g)D_{jm}^8(g)D_{kn}^8(g)d_{lmn},
\end{align*}
\]

and so \( f_{ijk} \) and \( d_{ijk} \) are invariant tensors in the same sense that \( \delta_{ij} \) and \( \epsilon_{i_1\ldots i_n} \) are invariant tensors for \( SO(n) \).

b) Let \( w_i = f_{ijk}u_jv_k \). Show that if \( u_i \rightarrow D_{ij}^8(g)u_j \) and \( v_i \rightarrow D_{ij}^8(g)v_j \), then \( w_i \rightarrow D_{ij}^8(g)w_j \). Similarly for \( w_i = d_{ijk}u_jv_k \). (Hint: show first that the \( D^8 \) matrices are real and orthogonal.) Deduce that \( f_{ijk} \) and \( d_{ijk} \) are Clebsh-Gordan coefficients for the \( 8 \oplus 8 \) part of the decomposition

\[
8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27.
\]

a) Similarly show that \( \delta_{\alpha\beta} \) and the lambda matrices \( (\lambda_i)_{\alpha\beta} \) can be regarded as Clebsch-Gordan coefficients for the decomposition

\[
3 \otimes \overline{3} = 1 \oplus 8.
\]

d) Use the graphical method, introduced in class, of plotting weights and pealing off irreps to obtain the tensor product decomposition in part b).