1) **Infinitesimal Homotopy:** Use the infinitesimal homotopy relation to show that the Lie derivative $\mathcal{L}$ commutes with the exterior derivative $d$, i.e. for $\omega$ a $p$-form, we have

$$d(\mathcal{L}_X \omega) = \mathcal{L}_X (d\omega).$$

2) **Magnetic solid:** The semi-classical dynamics of charge $-e$ electrons in a magnetic solid are governed by the equations

$$\dot{r} = \frac{\partial \epsilon(k)}{\partial k} - \hat{k} \times \Omega,$$

$$\dot{k} = -\frac{\partial V}{\partial r} - e \dot{r} \times B.$$

Here $k$ is the Bloch momentum of the electron, $r$ is its position, $\epsilon(k)$ its band energy (in the extended-zone scheme), and $B(r)$ is the external magnetic field. The components $\Omega_i$ of the *Berry curvature* $\Omega(k)$ are given in terms of the periodic part $|u(k)\rangle$ of the Bloch wavefunctions of the band by

$$\Omega_i(k) = i \epsilon_{ijk} \frac{1}{2} \left( \left\langle \frac{\partial u}{\partial k_j} \frac{\partial u}{\partial k_k} \right| - \left| \frac{\partial u}{\partial k_k} \frac{\partial u}{\partial k_j} \right\rangle \right).$$

The only property of $\Omega$ needed for the present problem, however, is that $\text{div}_k \Omega = 0$.

a) Show that these equations are Hamiltonian, with

$$H(r, k) = \epsilon(k) + V(r)$$

and

$$\omega = dk_i dx_i - e \epsilon_{ijk} B_i(r) dx_j dx_k + \frac{1}{2} \epsilon_{ijk} \Omega_i(k) dk_j dk_k.$$

as the symplectic form.

b) Confirm that the $\omega$ defined in part b) is closed, and that the Poisson brackets are given by

$$\{x_i, x_j\} = \frac{\epsilon_{ijk} \Omega_k}{(1 + eB \cdot \Omega)};$$

$$\{x_i, k_j\} = -\frac{\delta_{ij} + e \Omega_i B_j}{(1 + eB \cdot \Omega)};$$

$$\{k_i, k_j\} = \frac{\epsilon_{ijk} e B_k}{(1 + eB \cdot \Omega)}.$$
c) Show that the conserved phase-space volume $\omega^3/3!$ is equal to

$$(1 + eB \cdot \Omega) d^3k d^3x,$$

instead of the textbook $d^3k d^3x$.

3) **Non-abelian gauge fields as matrix-valued forms**: In a non-abelian gauge theory, such as QCD, the vector potential

$$A = A_\mu dx^\mu$$

becomes matrix-valued, meaning that the components, $A_\mu$, are matrices that do not necessarily commute with each other. The matrix-valued field-strength $F$ is a 2-form defined by

$$F = dA + A^2 = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu.$$

Here, a combined matrix and wedge product is to be understood:

$$(A^2)_{ik} \equiv \sum_j A_{ij} \wedge A_{jk} = \sum_j A_{ij;\mu} A_{jk;\nu} dx^\mu dx^\nu.$$

i) Show that $A^2 = \frac{1}{2} [A_\mu, A_\nu] dx^\mu dx^\nu$, and hence show that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

ii) Define *gauge-covariant derivatives*

$$\nabla_\mu = \partial_\mu + A_\mu,$$

and show that the commutator of two of these is equal to

$$[\nabla_\mu, \nabla_\nu] = F_{\mu\nu}.$$

iii) Let $g$ be an invertible matrix, and $\delta g$ a matrix describing a small change in $g$. Show that the corresponding change in the inverse matrix is given by $\delta(g^{-1}) = -g^{-1}(\delta g)g^{-1}$.

iv) Show that a necessary condition for the matrix-valued gauge field $A$ to be “pure gauge”, *i.e.* for there to be a position dependent matrix $g$ such that $A = g^{-1}dg$, is that $F = 0$.

v) Show that under the *gauge transformation*

$$A \rightarrow A^g \equiv g^{-1}Ag + g^{-1}dg,$$

we have $F \rightarrow g^{-1}Fg$. (Hint: The labour is minimized by exploiting the covariant derivative identity in ii)).

vi) Show that $F$ obeys the *Bianchi identity*

$$dF - FA + AF = 0.$$

This equation is the non-abelian version of the source-free Maxwell equations.
vii) Show that, in any number of dimensions, the Bianchi identity implies that the 4-form $\text{tr} (F^2)$ is closed, *i.e.* that $d \text{tr} (F^2) = 0$. (The trace is being taken only over the matrix indices.)

viii) Show that,

$$\text{tr} (F^2) = d \left\{ \text{tr} (A dA + \frac{2}{3} A^3) \right\},$$

so that if $\int_{\Omega} \text{tr} (F^2) \neq 0$, and $\partial \Omega = \emptyset$, then there cannot be a globally-defined $A$ on the region $\Omega$. The 3-form $\text{tr} (A dA + \frac{2}{3} A^3)$ is called a *Chern-Simons form.*

When the gauge group is $\text{SU}(n)$, the integral

$$c_2(A) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{tr} (F^2)$$

is an integer-valued topological invariant called the *Chern number*, or *instanton number*, of the gauge field configuration $A$.

The $2n$-forms $\text{tr} (F^n)$ are also closed, and can locally be written as the $d$ of $(2n - 1)$-form generalizations of the Chern-Simons form.