Q1 Binomial Series: Show that the binomial series expansion of \((1 + x)^{-\nu}\) can be written as
\[
(1 + x)^{-\nu} = \sum_{m=0}^{\infty} (-x)^m \frac{\Gamma(m + \nu)}{\Gamma(\nu) \, m!}, \quad |x| < 1.
\]

Q2 A Mellin transform and its inverse: Combine the Beta-function identity with a suitable change of variables to evaluate the Mellin transform
\[
\int_{0}^{\infty} x^{s-1}(1 + x)^{-\nu} \, dx, \quad \nu > 0,
\]
of \((1 + x)^{-\nu}\) as a product of Gamma functions. Now consider the Bromwich contour integral
\[
\frac{1}{2\pi i \Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(\nu - s)\Gamma(s) \, ds.
\]
Here \(\text{Re} \, c \in (0, \nu)\). The contour therefore runs parallel to the imaginary axis with the poles of \(\Gamma(s)\) to its left and the poles of \(\Gamma(\nu - s)\) to its right. Use the identity
\[
\Gamma(s)\Gamma(1 - s) = \pi \csc \pi s
\]
to show that when \(|x| < 1\) the contour can be closed by a large semicircle lying to the left of the imaginary axis. By using the preceding exercise to sum the contributions from the enclosed poles at \(s = -n\), evaluate the integral and so verify that the Bromwich contour provides the inverse of the Mellin transform in this case.

Q3 Mellin-Barnes integral.: Use the technique developed in the preceding exercise to show that
\[
F(a, b, c; -x) = \frac{\Gamma(c)}{2\pi i \Gamma(a)\Gamma(b)} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{\Gamma(a - s)\Gamma(b - s)\Gamma(s)}{\Gamma(c - s)} \, ds,
\]
for a suitable range of \(x\). This integral representation of the hypergeometric function is due to the English mathematician Ernest Barnes (1908), later a controversial Bishop of Birmingham.

Q4 Conformal block equation: Let
\[
Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]
Show that the matrix differential equation

\[ \frac{d}{dx} Y = \frac{A}{z} Y + \frac{B}{1-z}, \]

where \( A = \begin{pmatrix} 0 & a \\ 0 & 1-c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b & a+b-c+1 \end{pmatrix}, \)

has a solution

\[ Y(z) = F(a, b, c, z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{z}{a} F'(a, b; c, z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

(This result is useful in conformal field theory)