1 Infinitesimal Homotopy

The infinitesimal homotopy relation states

\[ \mathcal{L}_X \omega = (d i_X + i_X d) \omega. \]  

(1)

Taking the exterior derivative of \( \mathcal{L}_X \omega \),

\[ d(\mathcal{L}_X \omega) = d(d i_X + i_X d) \omega \]

(using (1))

\[ = d^2 i_X \omega + d(i_X d \omega) \]

\[ = (d i_X + i_X d)(d \omega) \] (add \( i_X d^2 \omega = 0 \))

\[ = \mathcal{L}_X (d \omega). \]

2 Magnetic Solid

(a) We need to verify that

\[ \dot{x} = \frac{\partial \epsilon(k)}{\partial k} - \dot{k} \times \Omega \]  

(2a)

\[ \dot{k} = -\frac{\partial V}{\partial x} - e \dot{x} \times B \]  

(2b)

is indeed the Hamiltonian vector flow of \( H(x, k) = \epsilon(k) + V(x) \) with the symplectic form \( \omega \). This amounts to checking that

\[ dH = -i_{v_H} \omega = -\omega(v_H, \cdot) \]  

(3)

reproduces equations (2a) and (2b). First, expand \( dH \) in local coordinates we find

\[ dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial k^i} dk^i = \frac{\partial V(x)}{\partial x^i} dx^i + \frac{\partial \epsilon(k)}{\partial k^i} dk^i. \]  

(4)

Next, plugging in the velocity vector field,

\[ v_H = \dot{x}^i \frac{\partial}{\partial x^i} + \dot{k}^i \frac{\partial}{\partial k^i}. \]  

(5)
into $-\omega(v_H, \cdot)$, we find

$$
-\omega(v_H, \cdot) = - \left\{ dk^i dx^i - \frac{e}{2} \epsilon_{ijk} B^j(x) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \Omega^i(k) dk^j dk^k \right\} \left( \dot{x}^l \frac{\partial}{\partial x^l} + k^l \frac{\partial}{\partial k^l}, \cdot \right) 
$$

$$
= - \dot{k}_i dx^i + \dot{x}_i dk_i + \frac{e}{2} \epsilon_{ijk} B^j(x) \left[ \dot{x}_j dx^k - \dot{x}_k dx^j \right] - \frac{1}{2} \epsilon_{ijk} \Omega^i(k) \left[ \dot{k}_j dk^k - \dot{k}_k dk^j \right] 
$$

$$
= \left[ - \dot{k}_k - e \epsilon_{ijk} \dot{x}^j B^i(x) \right] dx^k + \left[ \dot{x}_k + \epsilon_{ijk} k^i \Omega^j(k) \right] dk^k, 
$$

where in the last line I’ve relabeled indices, $i \leftrightarrow j$, and used equation (4) to write the underset equalities. These equalities reproduce equations (2a) and (2b), as desired.

(b) We first check that $\omega$ is closed.

$$
d\omega = d \left\{ dk^i dx^i - \frac{e}{2} \epsilon_{ijk} B^j(x) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \Omega^i(k) dk^j dk^k \right\} 
$$

$$
= - \frac{e}{2} \epsilon_{ijk} \left( dB^i(x) \right) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} (d\Omega^i(k)) dk^j dk^k 
$$

$$
= - \frac{e}{2} \epsilon_{ijk} \left( \frac{\partial B^i(x)}{\partial x^l} dx^l \right) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \left( \frac{\partial \Omega^i(k)}{\partial k^l} dk^l \right) dk^j dk^k 
$$

$$
= - \frac{e}{2} \epsilon_{ijk} \left( \frac{\partial B^i(x)}{\partial x^l} \right) dx^j dx^k + \frac{1}{2} \epsilon_{ijk} \left( \frac{\partial \Omega^i(k)}{\partial k^l} \right) dk^j dk^k \quad \text{(antisymmetry} \implies \ell = i) .
$$

Now the product $\epsilon_{ijk} dx^i dx^j dx^k$ is just proportional to $dx^1 dx^2 dx^3$ (and likewise for the $dk^i dk^j dk^k$).

Hence we can write

$$
d\omega \propto - \frac{e}{2} \left( \frac{\partial B^i(x)}{\partial x^l} \right) dx^1 dx^2 dx^3 + \frac{1}{2} \left( \frac{\Omega^i(k)}{\partial k^l} \right) dk^1 dk^2 dk^3 .
$$

But this vanishes identically since $\text{div}_x B = \text{div}_k \Omega = 0$. Hence $d\omega = 0$, as desired.

To show the desired Poisson brackets, first we find expressions for $\dot{x}_i$ and $\dot{k}_i$ using equations (1) and (2). To this end, note the following dot product equalities,

$$
\mathbf{\text{1}} \implies \dot{x} \cdot \Omega = \frac{\partial e(k)}{\partial k} \cdot \Omega - ( \dot{k} \times \Omega ) \cdot \Omega = 0 \quad (6a) 
$$

$$
\mathbf{\text{2}} \implies \dot{k} \cdot B = - \frac{\partial V(x)}{\partial x} \cdot B - (e\dot{x} \times B) \cdot B. = 0 \quad (6b)
$$

\(^1\text{For notational convenience, I will drop the wedge product and write } dx^i dx^j \text{ in place of } dx^i \wedge dx^j. \text{ I will also simply write } dx^i \text{ rather than } dx^i( \cdot ), \text{ although it is still implied that the dual basis elements act on basis elements (of the tangent space).}\)
Plugging into \(2b\) into \(2a\),

\[
\dot{x} = \frac{\partial \epsilon (k)}{\partial k} - \left[ -\frac{\partial V(x)}{\partial x} - e\dot{x} \times B \right] \times \Omega
\]

\[
= \frac{\partial \epsilon (k)}{\partial k} + \frac{\partial V(x)}{\partial x} \times \Omega - \left[ e\dot{x} (\Omega \cdot B) - B (e \frac{\partial \epsilon (k)}{\partial k} \cdot \Omega) \right]
\]

(BAC-CAB Rule)

\[
= \frac{\partial \epsilon (k)}{\partial k} + \frac{\partial V(x)}{\partial x} \times \Omega - \left[ e\dot{x} (\Omega \cdot B) - B \left( e \frac{\partial \epsilon (k)}{\partial k} \cdot \Omega \right) \right]
\]

(equation (6a)).

This rearranges to

\[
\dot{x} (1 + eB \cdot \Omega) = \frac{\partial \epsilon (k)}{\partial k} + \frac{\partial V(x)}{\partial x} \times \Omega + B \left( e \frac{\partial \epsilon (k)}{\partial k} \cdot \Omega \right).
\]

(7)

The analogous procedure for \(\dot{k}\) yields

\[
\dot{k} (1 + eB \cdot \Omega) = -\frac{\partial V(x)}{\partial x} - e \frac{\partial \epsilon (k)}{\partial k} \times B - \Omega \left( e \frac{\partial V(x)}{\partial x} \cdot B \right).
\]

(8)

Now, in part (a) we showed that equations \(\dot{x}\) and \(\dot{\epsilon}\) are Hamiltonian with \(\omega\) as the symplectic form for any Hamiltonian of the form \(H(x, k) = \epsilon (k) + V(x)\). We can then easily relate the time derivatives of functions with the Poisson bracket with a Hamiltonian function via

\[
\{ H_1, H_2 \} \overset{\text{def}}{=} \frac{dH_2}{dt} \bigg|_{H_1} = \dot{H}_2.
\]

(9)

Or equivalently, \(\{ f, H \} = -\dot{f}\) (since \(\{ f, g \} = -\{ g, f \}\)).

The computation of the Poisson brackets follows immediately. Choosing \(f = x_i\) and \(H = x_j\), equation \(\dot{x}\) yields

\[
\{ x_i, x_j \} = -\dot{x}_i = -\frac{\epsilon_{ijk} \Omega_k}{1 + eB \cdot \Omega}.
\]

The remaining two Poisson brackets follow by the same procedure but with the Hamiltonian function \(H = k_j\). Summarizing, one finds

\[
\{ x_i, x_j \} = -\frac{\epsilon_{ijk} \Omega_k}{(1 + eB \cdot \Omega)}, \quad \{ x_i, k_j \} = -\frac{\delta_{ij} + eB_i \Omega_j}{(1 + eB \cdot \Omega)}, \quad \{ k_i, k_j \} = \frac{\epsilon_{ijk} eB_k}{(1 + eB \cdot \Omega)}.
\]

(c) The conserved phase-space volume \(\omega^3/3!\) can be computed by direct calculation. Note that terms like \(dx_i \wedge dx_j \wedge dx_k \wedge dx_l\) vanish since necessarily there will be one repeated index for

\[\text{See equation (11.96) in the textbook. Also note that the definition given here and in the textbook differs by a minus sign from the traditional one. The literature is sometimes inconsistent with which definition is used, so it is always worth checking the convention used.}\]
three spatial dimensions (an analogous argument holds for the $k$'s). Hence

$$
\omega^3 = \int dx^i dx^j dx^k dk^i dk^j dk^k + 3!(dk^i dx^i) \left( \frac{e}{2} \epsilon^{i'j'k'} B^{i'} dx^{i'} dx^{j'} dx^{k'} \right) \left( \frac{1}{2} \epsilon_{i''j''k''} \Omega^{i''} dk^{i''} \right)
$$

$$
= dk^i dk^j dx^i dx^j dk^k - 3! \left( e \epsilon^{i'j'k'} B^{i'} \right) \Omega^{i''} dk^i dk^j dx^i dx^j dx^k
$$

$$
= 3! \left[ 1 + (e B \cdot \Omega) \right] dk^i dk^j dx^i dx^j dx^k
$$

which implies $\omega^3/3! = (1 + e B \cdot \Omega) d^3k d^3x$, as desired.

### 3 Non-abelian Gauge Fields as Matrix-valued Forms

(i) Given $A = A_\mu dx^\mu$, write

$$
2A^2 = A_\mu A_\nu dx^\mu dx^\nu + A_\mu A_\nu dx^\mu dx^\nu = (A_\mu A_\nu - A_\nu A_\mu) dx^\mu dx^\nu
$$

$$
\implies A^2 = \frac{1}{2} [A_\mu, A_\nu] dx^\mu dx^\nu,
$$

where in the last equality I’ve used $dx^\mu dx^\nu = -dx^\nu dx^\mu$ and relabeled indices. Similarly, one finds

$$
dA = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu dx^\nu,
$$

so that

$$
F = dA + A^2 = \left( \partial_\mu \partial_\nu - \partial_\nu \partial_\mu + [A_\mu, A_\nu] \right) dx^\mu dx^\nu.
$$

(ii) Using the definition of the gauge-covariant derivatives,

$$
\nabla_\mu = \partial_\mu - A_\mu,
$$

one finds

$$
[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu
$$

$$
= (\partial_\mu - A_\mu)(\partial_\nu - A_\nu) - (\partial_\nu - A_\nu)(\partial_\mu - A_\mu)
$$

$$
= \partial_\mu \partial_\nu + (\partial_\mu A_\nu) + A_\nu \partial_\mu + A_\mu \partial_\nu + A_\mu A_\nu - \partial_\nu \partial_\mu - A_\mu \partial_\nu - A_\nu \partial_\mu - A_\mu A_\nu
$$

(expand and cancel)

$$
= \partial_\mu A_\nu - (\partial_\nu A_\mu) + A_\mu A_\nu - A_\nu A_\mu
$$

$$
= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
$$

$$
= F_{\mu\nu}
$$

(by part (i)).

(iii) Let $g$ be an invertible matrix and $\delta g$ be a matrix describing a small change in $g$ (we assume
\( g + \delta g \) is still invertible).

\[
(g + \delta g)(g^{-1} + \delta(g^{-1})) = \text{id} \implies (g + \delta g)(g^{-1} + (\delta g)g^{-1} + O(\delta^2)) = \text{id}
\]

\[
\implies (\delta g)^{-1} = -g^{-1}(\delta g)g^{-1}.
\]

Alternatively, by demanding there is no variation in the identity, we find
\[
0 = \delta(gg^{-1}) = (\delta g)g^{-1} + g\delta(g^{-1}) \implies \delta(g^{-1}) = -g^{-1}(\delta g)g^{-1}.
\]

(iv) Suppose that the matrix-valued gauge field is a “pure gauge”; i.e., that
\[
A \equiv g^{-1}dg.
\]

Then
\[
dA = d(g^{-1}dg) = d(g^{-1})dg = -g^{-1}dg g^{-1}dg = (g^{-1}dg)^{-1}.
\]

This shows that
\[
F = dA + A^2 = -(g^{-1}dg)^2 + (g^{-1}dg)^2 = 0,
\]

as desired.

(v) Under a gauge transformation,
\[
A_\mu \mapsto A_\mu^g \equiv g^{-1}A_\mu g + g^{-1}(\partial_\mu g).
\]

Therefore, the covariant derivative transforms like
\[
\nabla_\mu \mapsto \nabla_\mu^g \equiv \partial_\mu + A_\mu^g = g^{-1}g\partial_\mu + g^{-1}(\partial_\mu g) + g^{-1}A_\mu g = g^{-1}(\partial_\mu + A_\mu)g.
\]

In the last equality, we have used the fact that the derivative acts to the right along with the chain rule (in reverse). Hence, \( \nabla_\mu \mapsto g^{-1}\nabla_\mu g \) under a gauge transformation. Using the result from part (ii) \( F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \), we can easily find how \( F_{\mu\nu} \) behaves when transformed.

\[
F_{\mu\nu}^g = [\nabla_\mu^g, \nabla_\nu^g] = g^{-1}[\nabla_\mu, \nabla_\nu]g = g^{-1}F_{\mu\nu}g,
\]

as desired.

(vi) To show the Bianchi identity, we simply take the exterior derivative of \( F \).

\[
dF = d(dA + A^2)
\]

\[
= d^2A + (dA)A - A(dA)
\]

\[
= (F - A^2)A - A(F - A^2)
\]

\[
= FA - AF.
\]

This rearranges to \( dF - FA + AF = 0 \), as desired.
(vii) Next, use the Bianchi identity to show that the 4-form is closed.

\[
d \text{tr}(F^2) = \text{tr}(dF^2) = \text{tr}((dF)F + F(dF)) = \text{tr}((FA - AF)F + F(FA - AF)) = \text{tr}(EAFF - AFF + FFA - EAF) = -\text{tr}(AFF) + \text{tr}(AFF) = 0.
\]

Note that in this case the cyclic permutation of matrix-valued forms is also even so that this operation doesn’t change the sign in the wedge product.

\[
\text{tr}(FFA) = \text{tr}(F_{\mu\nu}F_{\gamma\delta}A_{\lambda})dx^\mu dx^\nu dx^\gamma dx^\delta = (-1)^4 \text{tr}(A_\lambda F_{\mu\nu}F_{\gamma\delta})dx^\lambda dx^\mu dx^\nu dx^\gamma dx^\delta = \text{tr}(AFF).
\]

(viii) Before showing that

\[
\text{tr}(F^2) = d \left\{ \text{tr} \left( AdA + \frac{2}{3} A^3 \right) \right\},
\]

first note that \( \text{tr}(A^4) = 0 \) since

\[
\text{tr}(A^4) = \text{tr}(A_\mu A_\nu A_\gamma A_\delta)dx^\mu dx^\nu dx^\gamma dx^\delta = (-1)^3 \text{tr}(A_\delta A_\mu A_\nu A_\gamma)dx^\delta dx^\mu dx^\nu dx^\gamma = -\text{tr}(A^4).
\]

In the second equality, we have cyclically permuted matrices under the trace; however, unlike the product of three matrices in part (vii), this is obtained via an odd permutation which introduces a minus in the 4-form. With this, we can show beginning from the right-hand-side.

\[
d \left\{ \text{tr} \left( AdA + \frac{2}{3} A^3 \right) \right\} = \text{tr} \left\{ (dA)^2 + \frac{2}{3} [(dA)A^2 + A(dA)A + A^2(dA)] \right\} = \text{tr} \left\{ (dA)^2 + A^2(dA) + (dA)A^2 + (A^2)^2 \right\} = \text{tr} \left\{ (dA + A^2)^2 \right\} = \text{tr} \left\{ F^2 \right\}.
\]

Hence \( \text{tr}(F^2) \) is also exact. In the first line, I’ve used the cyclic property of the trace to write \(^3\)Again, the wedge product is implicit.
$A(dA)A$ in a symmetric way.