1 Lobachevski Space

There are several ways to do this problem. Various possible solutions are shown below.

**Method 1:** (Mike’s solution)

If we take $R = 1$ the point P has coordinates

$$X = \sinh s, \quad Z = \cosh s,$$

where, in the geometry of Lorentz boosts, $s$ would be the *rapidity*. We can use the hyperbolic version

$$\sinh s = \frac{2t}{1-t^2}, \quad \cosh s = \frac{1+t^2}{1-t^2}$$

of the $t$-substitution. This satisfies $\cosh^2 s - \sinh^2 s = 1$ as it should. The geometry of the figure, followed by a line of algebra, shows that the tangent of the angle between the line QP and the Z axis is

$$\frac{\sinh s}{1+\cosh s} = t.$$

Thus $t$ has the geometric interpretation of being the radial distance in the $X,Y$ plane from the origin to point Q.
The Minkowski arc length is
\[ dX^2 - dZ^2 = (d \sinh s)^2 - (d \cosh s)^2 = (\cosh^2 s - \sinh^2 s) ds^2 = ds^2 \]
so \( ds \) plays the role on the unit Minkowski hyperbola as \( d\theta \) on the unit circle. From

\[ \sinh s = \frac{2t}{1 - t^2} \]

we read off that

\[ (\cosh s) ds = \frac{2(1 + t^2)}{(1 - t^2)^2} dt \]
or

\[ ds = \frac{2}{1 - t^2} dt. \]

Thus for radial displacements

\[ ds^2 = \frac{4}{1 - t^2} dt^2 = \frac{4}{1 - X^2 + Y^2} dt^2 = \frac{4}{1 - X^2 + Y^2} (dX^2 + dY^2). \]

As \( dX^2 + dY^2 = dt^2 + t^2 d\phi^2 \) and for angular displacements \( ds^2 = \sinh^2 s d\phi^2 \) the formula is correct in that case also.

**Method 2:** One can also follow analogous steps to what was done in the “Stereographic Projection” problem from the previous homework set. Using hyperbolic polar coordinates,

\[
\begin{align*}
  x(\theta, \phi) &= R \cos \phi \sinh \theta \\
  y(\theta, \phi) &= R \sin \phi \sinh \theta \\
  z(\theta, \phi) &= R \cosh \theta,
\end{align*}
\]

the metric is \( g(\cdot, \cdot) = d\phi \otimes d\phi + \sinh^2 \theta d\theta \otimes d\theta \). This can then be mapped to the Poincaré disk model via the transformation \( \zeta = X + iY = e^{i\phi} \tanh(\theta/2) \). Following identical steps to the computation performed in the previous homework (i.e., compute the Jacobian, then use it to transform \( g \), which is just a doubly covariant tensor), one finds the new induced metric to be

\[
\frac{4R^4}{(R^2 - X^2 - Y^2)^2} (dX \otimes dX + dY \otimes dY).
\]

Note that here \( X \) and \( Y \) are coordinates on the Poincaré disk (in the previous problem set, the analogous variables were named \( \xi \) and \( \eta \)) whereas \( x, y, \) and \( z \) are coordinates on the upper half hyperboloid.

**Method 3:** Another “brute force” procedure one might follow is to start with the stereographic
projection,

\[
X(x, y) = R \left( \frac{2Rx}{R^2 + x^2 + y^2} \right), \\
Y(x, y) = R \left( \frac{2Ry}{R^2 + x^2 + y^2} \right), \\
Z(x, y) = R \left( \frac{-R^2 + x^2 + y^2}{R^2 + x^2 + y^2} \right),
\]

where \(\{X, Y, Z\}\) are the coordinates on \(S^2\) and \(\{x, y\}\) are the coordinates in the plane, and then plug in an imaginary radius (i.e., take \(R \mapsto iR\) in the above mapping) as suggested in the problem. The induced metric is just that of the Poincaré disk model. The computation can be performed easily in Mathematica.

\[
\ln[1] = \text{(* stereographic projection *)} \\
\text{SetAttributes[R, Constant];} \\
X[x, y] := R \left( \frac{2Rx}{R^2 + (x^2 + y^2)} \right); \\
Y[x, y] := R \left( \frac{2Ry}{R^2 + (x^2 + y^2)} \right); \\
Z[x, y] := R \left( \frac{-R^2 + (x^2 + y^2)}{R^2 + (x^2 + y^2)} \right); \\
\text{(* substitute in an “imaginary radius” and calculate metric *)} \\
Dt[X[x, y] /. R \mapsto iR]^2 + Dt[Y[x, y] /. R \mapsto iR]^2 + Dt[Z[x, y] /. R \mapsto iR]^2 // FullSimplify \\
\text{Out[5]=} \frac{4R^4 (\text{Dt}[x]^2 + \text{Dt}[y]^2)}{(-R^2 + x^2 + y^2)^2}
\]

2 Flywheel and Rolling Ball

(a) Here we work in the body-frame coordinates, with the (principle) \(Z\) axis along the direction of the axle. In these coordinates, the inertia tensor is diagonal and, as a result of the symmetry about the axle, \(I_{XX} = I_{YY}\). Since there are no external torques, we have that \(L_Z = I_{ZZ} \omega_Z = I_{ZZ}(\dot{\psi} + \dot{\phi} \cos \theta)\) is a constant of motion\(^1\) When the axle has returned to rest in the initial position, we have \(L_Z = 0\); hence, \(\dot{\psi} = -\dot{\phi} \cos \theta\) at all points on the curve \(\gamma = \partial \Omega\). Integrating this over the time required to make a closed loop, we find

\[
\Delta \psi = -\int_0^\tau \dot{\phi}(t) \cos \theta(t) \, dt
\]

\(^1\)One can also see this via the Lagrangian and the Euler-Lagrange equations.
\[
\begin{align*}
&= -\int_{\partial\Omega} \cos \theta(\phi) \, d\phi \quad \text{(parametrize } \theta \text{ in terms of } \phi) \\
&= -\int_{\Omega} d(\cos \theta \, d\phi) \quad \text{(Stokes' Theorem)} \\
&= \int_{\Omega} \sin \theta \, d\theta \wedge d\phi \\
&= \text{Area}(\Omega).
\end{align*}
\]

Notice that if we reverse the orientation of the path, then the enclosed area becomes \(4\pi - \text{Area}(\Omega)\). Since reversing orientation changes the sign, we have that \(4\pi - \text{Area}(\Omega) = -\text{Area}(\Omega)\), which shows the area is only defined modulo \(4\pi\).

(b) Since the point in contact with the table describes a closed path on the ball, we instead use \textit{space-fixed coordinates}\(^2\) so that \(\omega_Z = \dot{\phi} + \dot{\psi} \cos \theta\), and the no slip condition implies \(\dot{\phi} + \dot{\psi} \cos \theta = 0\).\(^3\) Analogous steps to those of part (a) show that \(\Delta \phi = \text{Area}(\Omega)\).

3 Hopf Invariant

Before delving into calculations, it is worth summarizing some of the notation and identities we make use of throughout the solution. Given in the problem, we have

\[
\begin{align*}
\frac{Dv}{Dt} &\equiv \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla P \\
\nabla \cdot v &= 0
\end{align*}
\]

(Euler’s equation) \hspace{1cm} \text{(1)}

\hspace{1cm} \text{(incompressibility condition).} \hspace{1cm} \text{(2)}

We also use the following vector calculus identities, which are easily proved by writing terms out in index notation\(^4\)

\[
\begin{align*}
\nabla \cdot (\psi A) &= (\nabla \psi) \cdot A + \psi(\nabla \cdot A) \\
\nabla (A \cdot B) &= \nabla_A (A \cdot B) + \nabla_B (A \cdot B) \\
A \times (\nabla \times B) &= \nabla_B (A \cdot B) - (A \cdot \nabla) B \\
\nabla \times (A \times B) &= A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla) A - (A \cdot \nabla) B,
\end{align*}
\]

where I’ve used Feynman’s subscript notation, \(\nabla_A (A \cdot B) \equiv B_k(\partial_j A_k)\hat{e}_j\), to denote the gradient acts only on the vector in the subscript.

(a) \hspace{1cm} (i) First note in equation (??), when \(A = B = v\), the \(\nabla_B (A \cdot B)\) can be written as \(\nabla (\frac{1}{2} v^2)\).

We can therefore write the curl of the convective derivative, \(\nabla \times \frac{Dv}{Dt} = \frac{D}{Dt} (\nabla \times v) = \frac{D\omega}{Dt}\).

\(^2\) Note that the expression for the angular velocity vector differs in space-fixed and body-fixed coordinates (see here).

\(^3\) This is the coordinate system employed in question 3 of homework 2 where we calculated the vector fields corresponding to the motions of a rolling ball.

\[
\frac{D\omega}{Dt} = \nabla \times \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right]
\]
\[
= \frac{\partial}{\partial t}(\nabla \times \mathbf{v}) + \nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v}]
\]
\[
= \frac{\partial \omega}{\partial t} + \nabla \times \left[ \nabla \left( \frac{1}{2} \mathbf{v}^2 \right) - \mathbf{v} \times \omega \right] = 0
\]
\[
= \frac{\partial \omega}{\partial t} + \nabla \times \left[ \nabla \left( \frac{1}{2} \mathbf{v}^2 \right) \right] - \nabla \times (\mathbf{v} \times \omega) \quad \text{(curl of gradient vanishes)}.
\]

Now expanding the remaining term using (??), we find
\[
-\nabla \times (\mathbf{v} \times \omega) = -\mathbf{v} (\nabla \cdot \omega) + \omega (\nabla \cdot \mathbf{v}) - (\omega \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\omega.
\]

Plugging this in, one finds
\[
\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{v} = \nabla \times (-\nabla P), = 0
\]
which re-arranges to
\[
\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{v}, \quad \text{(7)}
\]
as desired.

(ii) Using the product rule and plugging in equations (??) and (??) we find
\[
\frac{D}{Dt}(\mathbf{v} \cdot \omega) = \left[ \frac{D\mathbf{v}}{Dt} \right] \cdot \omega + \mathbf{v} \cdot \left[ \frac{D\omega}{Dt} \right]
\]
\[
= [-\nabla P] \cdot \omega + \mathbf{v} \cdot [(\omega \cdot \nabla)\mathbf{v}]
\]
\[
= \omega \cdot \left[ -\nabla P + \nabla \left( \frac{1}{2} \mathbf{v}^2 \right) \right] \quad \text{(equation (??), } \nabla \cdot \omega = 0). \]
\[
= \nabla \cdot \left[ \omega \left( \frac{1}{2} \mathbf{v}^2 - P \right) \right]
\]
\[
= \omega \cdot \left( \frac{1}{2} \mathbf{v}^2 - P \right)
\]
\]

(iii) For a volume \( \Omega(t) \) that is co-moving with a fluid (and is allowed to change shape), we need some kind of generalization of Leibniz’s integral rule. In the three dimensional case, the appropriate generalization is known as the Reynolds Transport Theorem, which for

\footnote{Generalizations to higher dimensions can be clearly stated in the language of differential forms. See later comments for problem 4.}
incompressible fluids takes the form

\[
\frac{d}{dt} \int_{\Omega(t)} f(x, t) \, dV = \int_{\Omega(t)} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) f(x, t) \, dV. \tag{8}
\]

This is the exact statement we are asked to check in the problem. To prove this, consider a parametrized family of diffeomorphisms, \( \varphi_t : \Omega_0 \to \Omega(t) \) such that \( \varphi_t : \mathbf{u} \mapsto x \), which maps the region \( \Omega_0 \equiv \Omega(t = 0) \) to the corresponding region after it has been carried along the vector field \( \mathbf{v} \) for some time \( t \). Pulling back by this function, we can write

\[
\int_{\Omega(t)} f \, dx_1 \, dx_2 \, dx_3 = \int_{\Omega_0} \varphi_t^* (f \, dV) = \int_{\Omega_0} f(x(u), t) |J| \, \frac{du^1 \, du^2 \, du^3}{(=dV_0)},
\]

where \( |J| \equiv \left| \det \left( \frac{\partial x}{\partial u} \right) \right| \). This essentially moves the time-dependence of the region of integration into the integrand. We can then use Leibniz’s rule to write

\[
\frac{d}{dt} \int_{\Omega_0} f \, |J| \, dV_0 = \int_{\Omega_0} \left[ \left( \frac{\partial}{\partial t} + \frac{\partial \mathbf{u}}{\partial t} \cdot \nabla \mathbf{u} \right) f \, |J| + f \left( \frac{\partial}{\partial t} |J| \right) \right] \, dV_0.
\]

Note that \( \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \) and \( \frac{\partial}{\partial t} |J| = 0 \) since

\[
\frac{d}{dt} \text{Vol}(\Omega(t)) = \frac{d}{dt} \int_{\Omega(t)} dV = \frac{d}{dt} \int_{\Omega_0} |J| \, dV_0 = \int_{\Omega_0} \left( \frac{d}{dt} |J| \right) \, dV_0,
\]

so that the incompressibility condition \( \frac{d}{dt} \text{Vol}(\Omega(t)) = 0 \) implies that \( \frac{d}{dt} |J| = 0 \). After changing back to the original variables and placing the time-dependence back into the integration region, one finds

\[
\frac{d}{dt} \int_{\Omega(t)} f(x, t) \, dV = \int_{\Omega(t)} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) f(x, t) \, dV = \int_{\Omega(t)} \frac{Df}{Dt} \, dV,
\]

as desired.

(iv) Utilizing the results of the previous parts,

\[
\frac{d}{dt} H = \frac{d}{dt} \int \mathbf{v} \cdot \mathbf{\omega} \, dV
\]

\[
= \int \frac{D}{Dt} (\mathbf{v} \cdot \mathbf{\omega}) \, dV \quad \text{(part (iii))}
\]

\[
= \int \nabla \cdot \left\{ \mathbf{\omega} \left( \frac{1}{2} - P \right) \right\} \, dV \quad \text{(part (ii))}
\]

\[
= \int \left\{ \mathbf{\omega} \left( \frac{1}{2} - P \right) \right\} \cdot dS \quad \text{(Gauss’s law)}
\]

\[\text{Notice that here we need the absolute values on the Jacobian because we are considering the unoriented integral. In the final question, we will perform a similar calculation in the language of differential forms were the integrals are oriented. There the Jacobian factor is included with no absolute values.}\]
(b) (i) Since the electromotive force must vanish everywhere,
\[ E + v \times B = -\frac{\partial A}{\partial t} - \nabla \phi + v \times (\nabla \times A) = 0 \implies \frac{\partial A}{\partial t} = v \times (\nabla \times A) - \nabla \phi. \]

Utilizing the previous result, we have
\[ E = -\frac{\partial A}{\partial t} - \nabla \phi = -[v \times (\nabla \times A) - \nabla \phi] - \nabla \phi = -\nabla \times A = -v \times B. \]

Plugging this into Faraday’s law, \( \nabla \times E = -\frac{\partial B}{\partial t} \), yields \( \frac{\partial B}{\partial t} = \nabla \times (v \times B) \), as desired.

(ii) In analogy with the calculation in (a)(ii), we use the product rule to write \( \frac{D}{Dt}(A \cdot B) = \frac{\partial A}{\partial t} \cdot B + A \cdot \frac{\partial B}{\partial t} \). Using the identities in the previous part, each of the convective derivatives can be written as
\[
\begin{align*}
\frac{DA}{Dt} &= \frac{\partial A}{\partial t} + (v \cdot \nabla)A \\
&= v \times (\nabla \times A) - \nabla \phi + (v \cdot \nabla)A \\
&= \nabla_A(A \cdot v) - \nabla \phi \\
&= (\text{equation (?)})
\end{align*}
\]
and
\[
\begin{align*}
\frac{DB}{Dt} &= \frac{\partial B}{\partial t} + (v \cdot \nabla)B \\
&= \nabla \times (v \times B) + (v \cdot \nabla)B \\
&= v (\nabla \cdot B) - B (\nabla \cdot v) + (B \cdot \nabla)v \\
&= 0, \text{ incompressible}
\end{align*}
\]

Using these one finds
\[
\begin{align*}
\frac{D}{Dt}(A \cdot B) &= [\nabla_A(A \cdot v) - \nabla \phi] \cdot B + A \cdot (B \cdot \nabla)v \\
&= [\nabla_A(A \cdot v) - \nabla \phi] \cdot B + [\nabla_v(A \cdot v)] \cdot B \\
&= B \cdot [\nabla(A \cdot v) - \nabla \phi] \\
&= \nabla \cdot [B(A \cdot v - \phi)] \\
&= \nabla \cdot [B(\nabla \phi) - \nabla \phi] \\
&= \nabla \cdot [B(\nabla \phi)] \quad \text{(using (?) and } \nabla \cdot B = 0),
\end{align*}
\]
as desired.

(iii) This is analogous to the calculation in the preceding part. Putting all of the pieces together one finds
\[
\frac{d}{dt}W = \frac{d}{dt} \int_{\Omega} (A \cdot B) \, dV
\]
\[
\begin{align*}
\int_\Omega \frac{D}{Dt}(A \cdot B) \, dV &= \int_\Omega \nabla \cdot \{B (A \cdot v - \phi)\} \, dV \quad \text{(by (b)(ii))} \\
\int_{\partial \Omega} \{B(A \cdot v - \phi)\} \cdot dS &= \int_\Omega \nabla \cdot \{B (A \cdot v - \phi)\} \, dV \quad \text{(Gauss's law).}
\end{align*}
\]

If \( B \) vanishes at spatial infinity, then \( \frac{d}{dt} W = 0 \), which shows that \( W \) is a constant of motion.

4 Faraday’s Law

(a) Following analogous steps to the procedure done in 3(a)(iii) (but here written explicitly in the language of differential forms), we pull back the time-varying region of integration to one that is fixed, \( \Omega_0 \equiv \Omega(\tau = 0) \), via a diffeomorphism \( \varphi_t \).

\[
\begin{align*}
\frac{d}{d\tau} \int_{\Omega(\tau)} F &= \frac{d}{d\tau} \int_{\Omega(\tau)} \left( \frac{1}{2} F_{\mu\nu}(x) \, dx^\mu \wedge dx^\nu \right) \\
&= \frac{d}{d\tau} \int_{\varphi_t^{-1}(\Omega(\tau))} \varphi_t^* \left( \frac{1}{2} F_{\mu\nu}(x) \, dx^\mu \wedge dx^\nu \right) \\
&= \frac{d}{d\tau} \int_{\Omega_0} \left( \frac{1}{2} F_{\mu\nu}(x(\xi)) \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{\partial x^{\nu}}{\partial \xi^\rho} \, d\xi^\sigma \wedge d\xi^\rho \right) \\
&= \frac{1}{2} \int_{\Omega_0} \left( \frac{d}{d\tau} \left( F_{\mu\nu}(x(\xi)) \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{\partial x^{\nu}}{\partial \xi^\rho} \right) \right) d\xi^\sigma \wedge d\xi^\rho \\
&\quad + F_{\mu\nu}(x(\xi)) \left( \frac{d}{d\tau} \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{\partial x^{\nu}}{\partial \xi^\rho} + F_{\mu\nu}(x(\xi)) \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{d}{d\tau} \frac{\partial x^{\nu}}{\partial \xi^\rho} \right) d\xi^\sigma \wedge d\xi^\rho.
\end{align*}
\]

Notice however that

\[
\frac{\partial}{\partial \tau} \frac{\partial x^\mu}{\partial \xi^\sigma} = \frac{\partial}{\partial \xi^\sigma} \left( \frac{\partial x^\mu}{\partial \tau} \right) = \frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial}{\partial x^\lambda} V^\mu,
\]

and analogously for the other terms. We can therefore write

\[
\begin{align*}
\frac{d}{d\tau} \int_{\Omega(\tau)} F &= \frac{1}{2} \int_{\Omega_0} \left[ V^\lambda \frac{\partial F_{\mu\nu}}{\partial x^\lambda} \left( \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{\partial x^{\nu}}{\partial \xi^\rho} \right) \right. \\
&\quad + F_{\mu\nu} \frac{\partial V_{\nu}}{\partial x^\lambda} \left( \frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial x^\mu}{\partial \xi^\rho} \right) + F_{\mu\nu} \frac{\partial V_{\mu}}{\partial x^\lambda} \left( \frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial x^\mu}{\partial \xi^\rho} \right) \left. \right] d\xi^\sigma \wedge d\xi^\rho \\
&= \frac{1}{2} \int_{\Omega(\tau)} \left[ V^\lambda \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + F_{\nu\lambda} \frac{\partial V_{\nu}}{\partial x^\lambda} + F_{\mu\lambda} \frac{\partial V_{\mu}}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{\partial x^\nu}{\partial \xi^\rho} \right] d\xi^\sigma \wedge d\xi^\rho.
\end{align*}
\]
\[ \frac{1}{2} \int_{\Omega_0} \left[ V^\lambda \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + F_{\lambda\nu} \frac{\partial V^\lambda}{\partial x^\mu} + F_{\mu\lambda} \frac{\partial V^\lambda}{\partial x^\nu} \right] \, dx^\mu \wedge dx^\nu. \]

In the second equality we have simply re-labeled indices; in the last, the integrand has been written back in terms of the original coordinates with a time-varying region of integration. This shows that \( \frac{d}{d\tau} \int_{\Omega(\tau)} F = \int_{\Omega(\tau)} \mathcal{L}_\nu F \), as desired.

(b) If \( \tau \) is the proper time along the world-line of each element, then

\[ \frac{dV^\mu}{d\tau} = \frac{dt}{d\tau} \frac{dV^\mu}{dt} = \frac{1}{\sqrt{1 - v^2}} (1, v) \]

and

\[ f = -\nu F = - \left( \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \right) \left( V^\sigma \frac{\partial}{\partial x^\sigma} , \cdot \right) \]

\[ = - \frac{1}{2} F_{\mu\nu} \left( V^\sigma \delta^\mu_{\sigma} \, dx^\nu - V^\sigma \delta^\nu_{\sigma} \, dx^\mu \right) = F_{\mu\nu} V^\nu \, dx^\mu, \]

which is exactly the definition Lorentz-force 4-vector.

\[ ^7 \text{What we have done here is essentially derive Leibniz’s rule for 2-forms. Analogous results, which follow the same line of reasoning, can be derived for general } p \text{-forms. See } \text{[Flanders, Harley “Differentiation Under the Integral Sign”]} \text{ for a proof of the general statement.} \]