1) Lie Bracket Geometry: Consider the vector fields $X = y\partial_x$, $Y = \partial_y$ in $\mathbb{R}^2$. Find the flows associated with these fields, and use them to verify the statements made in the lecture about the geometric interpretation of the Lie bracket.

2) Frobenius' theorem: Show that the pair of vector fields $L_z = x\partial_y - y\partial_x$ and $L_y = z\partial_x - x\partial_z$ in $\mathbb{R}^3$ is in involution. Show further that the general solution of the system of partial differential equations

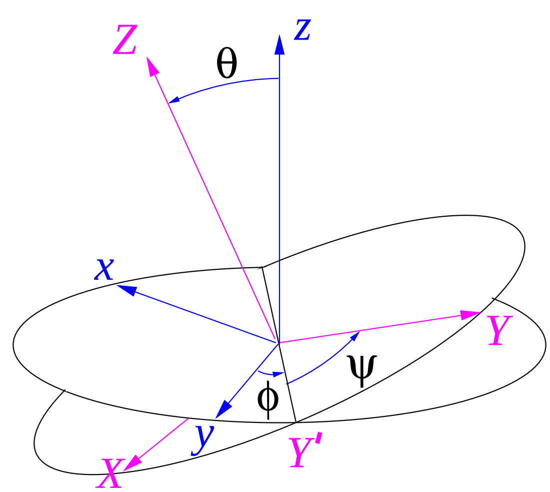
$$\begin{aligned}(x\partial_y - y\partial_x)f &= 0,\\ (x\partial_z - z\partial_x)f &= 0,\end{aligned}$$

in $\mathbb{R}^3$ is $f(x, y, z) = F(x^2 + y^2 + z^2)$, where F is an arbitrary function.

3) Rolling ball: In class we mentioned the rolling conditions for a ball on a table:

$$\begin{aligned}\dot{x} &= \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi,\\ \dot{y} &= -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi, \qquad (\star)\\ 0 &= \dot{\psi}\cos\theta + \dot{\phi}.\end{aligned}$$

Here, we are using the "Y" convention for Euler angles. In this convention θ and ϕ are the usual spherical polar co-ordinate angles with respect to the space-fixed xyz axes. They specify the direction of the body-fixed Z axis about which we make the final ψ rotation.



Euler angles: we first rotate the ball through an angle ϕ about the z axis, thus taking $y \to Y'$, then through θ about Y', and finally through ψ about Z, so taking $Y' \to Y$.

a) Show that ($\star$) are indeed the no-slip rolling conditions

$$\begin{aligned} \dot{x} &= \omega_y, \\ \dot{y} &= -\omega_x, \\ 0 &= \omega_z, \end{aligned}$$

where $(\omega_x, \omega_y, \omega_z)$ are the components of the ball's angular velocity in the xyz space-fixed frame.

b) Solve the three constraints ($\star$) so as to obtain the vector fields

$$\begin{aligned} \mathbf{roll_x} &= \partial_x - \sin\phi \cot\theta\, \partial_\phi + \cos\phi\, \partial_\theta + \operatorname{cosec}\theta \sin\phi\, \partial_\psi, \\ \mathbf{roll_y} &= \partial_y + \cos\phi \cot\theta\, \partial_\phi + \sin\phi\, \partial_\theta - \operatorname{cosec}\theta \cos\phi\, \partial_\psi. \end{aligned}$$

c) Show that

$$[\mathbf{roll_x}, \mathbf{roll_y}] = -\mathbf{spin_z},$$

where $\mathbf{spin_z} \equiv \partial_\phi$, corresponds to a rotation about a vertical axis through the point of contact. This is a new motion, being forbidden by the $\omega_z = 0$ condition.

d) Show that

$$\begin{aligned} [\mathbf{spin_z}, \mathbf{roll_x}] &= \mathbf{spin_x}, \\ [\mathbf{spin_z}, \mathbf{roll_y}] &= \mathbf{spin_y}, \end{aligned}$$

where the new vector fields

$$\begin{aligned} \mathbf{spin_x} &\equiv -(\mathbf{roll_y} - \partial_y), \\ \mathbf{spin_y} &\equiv (\mathbf{roll_x} - \partial_x), \end{aligned}$$

correspond to rotations of the ball about the space-fixed x and y axes through its centre, and with the centre of mass held fixed.

We have generated five independent vector fields from the original two. Therefore, by sufficient rolling to-and-fro, we can position the ball anywhere on the table, and in any orientation.

4) Killing Vector: The metric on the unit sphere equipped with polar co-ordinates is

$$g(\ ,\ ) = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi.$$

Consider

$$V_x = -\sin\phi \partial_\theta - \cot\theta \cos\phi \partial_\phi,$$

the vector field of a rigid rotation about the x axis. Compute the Lie derivative $\mathcal{L}_{V_x} g$, and show that it is zero.