Q1 Binomial Series: Show that the binomial series expansion of $(1 + x)^{-\nu}$ can be written as

$$(1 + x)^{-\nu} = \sum_{m=0}^{\infty} (-x)^m \frac{\Gamma(m + \nu)}{\Gamma(\nu) m!}, \quad |x| < 1.$$ 

Q2 A Mellin transform and its inverse: Combine the Beta-function identity with a suitable change of variables to evaluate the Mellin transform

$$\int_0^\infty x^{s-1}(1 + x)^{-\nu} \, dx, \quad \nu > 0,$$

of $(1 + x)^{-\nu}$ as a product of Gamma functions. Now consider the Bromwich contour integral

$$\frac{1}{2\pi i} \frac{\Gamma(\nu)}{\Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} x^{-s} \Gamma(\nu - s) \Gamma(s) \, ds.$$ 

Here $\text{Re } c \in (0, \nu)$. The contour therefore runs parallel to the imaginary axis with the poles of $\Gamma(s)$ to its left and the poles of $\Gamma(\nu - s)$ to its right. Use the identity

$$\Gamma(s) \Gamma(1 - s) = \pi \cot \pi s$$

to show that when $|x| < 1$ the contour can be closed by a large semicircle lying to the left of the imaginary axis. By using the preceding exercise to sum the contributions from the enclosed poles at $s = -n$, evaluate the integral and so verify that the Bromwich contour provides the inverse of the Mellin transform in this case.

Q3 Mellin-Barnes integral: Use the technique developed in the preceding exercise to show that

$$F(a, b, c; -x) = \frac{\Gamma(c)}{2\pi i} \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{\Gamma(a-s) \Gamma(b-s) \Gamma(s)}{\Gamma(c-s)} \, ds,$$

for a suitable range of $x$. This integral representation of the hypergeometric function is due to the English mathematician Ernest Barnes (1908), later a controversial Bishop of Birmingham.

Q4 Conformal block equation: Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
Show that the matrix differential equation

\[ \frac{d}{dx} Y = \frac{A}{z} Y + \frac{B}{1 - z} Y, \]

where

\[ A = \begin{pmatrix} 0 & a \\ 0 & 1 - c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b & a + b - c + 1 \end{pmatrix}, \]

has a solution

\[ Y(z) = F(a, b; c, z) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \frac{z}{a} F'(a, b; c; z) \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \]

(This result is useful in conformal field theory)