Solutions to Homework Set 11

Dielectric Sphere: The solution outside will be of the form

$$\begin{aligned}\Phi_{out}(r,\theta) &= -E_0 r P_1(\cos\theta) + \frac{C_1}{r^2} P_1(\cos\theta) \\ &= \left(-E_0 r + \frac{C}{r^2}\right)\cos\theta.\end{aligned}$$

while, inside

$$\Phi_{in}(r,\theta) = A_1 r P_1(\cos\theta).$$

These terms have been selected to match the asymptotically uniform electric field at infinity, to be finite at the centre of the sphere, and to all have the same angular dependence.

The continuity of the tangential electric field will be ensured if Φ is continuous at the surface of the sphere. The continuity of the radial component of $\mathbf{D}$ requires

$$\epsilon \frac{\partial \Phi_{in}}{\partial r} = \epsilon_0 \frac{\partial \Phi_{out}}{\partial r}$$

at $r = a$. These two conditions are

$$A_1 = -E_0 + \frac{C_1}{a^3},$$

and

$$\epsilon A_1 = -\epsilon_0 E_0 - 2\epsilon_0 \frac{C_1}{a^3},$$

respectively. This pair of equations may be solved to give

$$\begin{aligned}A_1 &= -\left(\frac{3\epsilon_0}{\epsilon + 2\epsilon_0}\right) E_0 \\ C_1 &= \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right) a^3 E_0.\end{aligned}$$

Therefore

$$\mathbf{E}_{in} = \left(\frac{3\epsilon_0}{\epsilon + 2\epsilon_0}\right) \mathbf{E}_0.$$

Hollow Sphere: The generating function for Legendre polynomials is

$$\frac{1}{(1+r^2-2r\cos\theta)^{1/2}} = \sum_{l=0}^{\infty} r^l P_l(\cos\theta), \quad r<1.$$

If we differentiate this relation with respect to r, and then multiply the result by r, we have

$$\frac{-r^2+r\cos\theta}{(1+r^2-2r\cos\theta)^{3/2}} = \sum_{l=0}^{\infty} l r^l P_l(\cos\theta).$$

Thus

$$\frac{-2r^2+2r\cos\theta+r^2+1-2r\cos\theta}{(1+r^2-2r\cos\theta)^{3/2}} = \sum_{l=0}^{\infty}(2l+1) r^l P_l(\cos\theta).$$

In other words

$$\frac{1-r^2}{(1+r^2-2r\cos\theta)^{3/2}} = \sum_{l=0}^{\infty}(2l+1) r^l P_l(\cos\theta), \quad r<1.$$

We use this in conjunction with the $r \le 1$ interior solution

$$\varphi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^l Y_m^l(\theta,\phi) \iint \left[Y_m^l(\theta',\phi')\right]^* \varphi(1,\theta',\phi')\, d\cos\theta' d\phi',$$

and the addition formula

$$(2l+1)P_l(\cos\gamma) = 4\pi \sum_{m=-l}^{l} Y_m^l(\theta,\phi)\left[Y_m^l(\theta',\phi')\right]^*,$$

with γ the angle between the directions (θ,ϕ) and (θ',ϕ'), to deduce that

$$\varphi(r,\theta,\phi) = \frac{1}{4\pi}\iint \frac{1-r^2}{(1+r^2-2r\cos\gamma)^{3/2}}\varphi(1,\theta',\phi')\, d\cos\theta' d\phi'.$$

For a sphere of radius a, we have, on dimensional grounds, that

$$\varphi(r,\theta,\phi) = \frac{a}{4\pi}\iint \frac{a^2-r^2}{(a^2+r^2-2r\cos\gamma)^{3/2}}\varphi(a,\theta',\phi')\, d\cos\theta' d\phi', \quad r \le a.$$

If we set $r=0$ we see that φ at any point is equal to the average over a small sphere surrounding that point, and so φ cannot have any local maxima or minima in the interior of any domain in which it satisfies $\nabla^2\varphi = 0$.

Rudolph Peierls' problem: This is the calculation that Heisenberg notoriously got wrong — his initial estimate for the critical mass of ^{235}U being about 40 tonnes, whereas in reality it is in the order of 10 kilogrammes. Having decided that an atomic bomb was not feasable, he concentrated on building a nuclear reactor using unenriched uranium metal. He then failed to realize that purified graphite could be used a moderator for this fuel – hence the demand for Norwegian heavy water. It remains a controversial question as to whether these mistakes influenced the outcome of WWII. The problem was correctly analysed by Peierls. The Peierls-Frisch memorandum that started the development of nuclear weapons can be read at: http://www.atomicarchive.com/Docs/FrischPeierls.shtml. You will find a much more detailed analysis of realistic critical mass calculations in an article by Jeremy Bernstein: American Journal of Physics, **70** (2002) 911. This paper can be read online if you are logging in from a UIUC address.

We solve our homework problem by observing that at the critical radius $R = R_{\rm crit}$ there will be a time-independent solution. For $R < R_{\rm crit}$ any intial neutron flux will decay with time, and for $R > R_{\rm crit}$ a single spontaneous fission will cause the assembly to explode. (What crucial phenomenon, ignored in our simplified mathematical model, makes real nuclear reactors controllable?)

Let us assume that $R = R_{\rm crit}$, so that we can forget about the time derivatives, whence

$$\begin{aligned} D_F\nabla^2 n + \nu n &= 0, \quad r < R \\ D_T\nabla^2 n &= 0, \quad r > R. \end{aligned}$$

The first mode to become unstable will be spherically symmetric, and so the relevant solutions are

$$n(r) = \begin{cases} j_0(kr), & r < R, \\ \text{const.}/r, & r > R. \end{cases}$$

Here $k^2 = \nu/D_F$. The compatibility condition at R (continuity of $n(r)$, together with $D_F\nabla n = D_T\nabla n$) is most compactly expressed as the continuity of Dn'/n. Thus

$$D_F k\frac{j_0'(kR)}{j_0(kR)} = -\frac{D_T}{R}.$$

Now $j_0(x) = (\sin x)/x$, and so the compatibility condition becomes

$$D_F k\left(\cot kR_{\rm crit} - \frac{1}{kR_{\rm crit}}\right) = -\frac{D_T}{R_{\rm crit}},$$

or, after tidying up,

$$kR_{\rm crit}\cot kR_{\rm crit} = 1 - \frac{D_T}{D_F}.$$

For $D_T = D_F$ we have $\cot kR_{\rm crit} = 0$, whose smallest solution gives $kR_{\rm crit} = \pi/2$. For $D_T = \infty$ (air), we have $\cot kR_{\rm crit} = \infty$, which makes $kR_{\rm crit} = \pi$. A uranium tamper therefore reduces the critical radius to one half of that with no tamper. Because the volume is proportional to $R_{\rm crit}^3$, the critical mass is reduced by a factor of eight.