Do all **three** *questions. Check each step before proceeding, there will be no propagation of errors. Marks will be* **subtracted** *for any equation that is obvious nonsense.*

1) Green function: Consider the *homogeneous* boundary value problem

$$-\frac{d^2y}{dx^2} = f(x), \quad x \in [0,1], \quad y'(0) = y(1) = 0.$$

a) Construct the explicit Green function appropriate to this problem. [10 points]
b) Use your Green function to write down the solution of the boundary value problem as the sum of two explicit integrals over complementary components of the unit interval. [10 points]
c) Confirm that your solution $y(x)$ obeys both boundary conditions, and by explicit differentiation confirm that it does indeed solve the original problem. [10 points]

Now consider the *inhomogeneous* boundary value problem

$$-y'' = f(x), \quad y'(0) = A, \quad y(1) = B.$$

d) Use the method based Lagrange's identity to obtain the solution to this boundary value problem. The points here are for exhibiting the *method*, so merely writing down the solution will not earn any credit. [10 points]

2) First Integral: In the course of solving the Brachistochrone problem we considered the functional

$$T[y] = \int_0^a \sqrt{\frac{1+y'^2}{2gy}}dx.$$

a) Write down and simplify the *first integral* for the corresponding Euler-Lagrange equation (You do *not* have to derive or write down the Euler-Lagrange equation itself). [10 points]
b) In class we solved the Euler-Lagrange equation by showing that it implies that

$$\frac{d}{dx}\left\{y(1+y'^2)\right\} = 0.$$

Use your first integral to verify this claim. [10 points]

3) Orthogonality and Completeness: The *Conical functions* $\varphi_\lambda(x)$ are the solutions to the differential equation

$$\frac{d}{dx}(x^2-1)\frac{d}{dx}\varphi_\lambda + (\lambda^2 + \tfrac{1}{4})\varphi_\lambda = 0$$

in the interval $[1, \infty]$ that obey the boundary condition $\varphi_\lambda(1) = 1$. The $\varphi_\lambda(x)$ are real-valued when λ^2 is real and positive, and $\varphi_\lambda(x) = \varphi_{-\lambda}(x)$. The $\varphi_\lambda(x)$ obey an orthogonality condition

$$\int_1^\infty \varphi_\lambda(x)\varphi_\mu(x)\,dx = \frac{1}{\lambda\tanh(\pi\lambda)}\delta(\lambda-\mu), \quad \lambda, \mu > 0,$$

The set $\{\varphi_\lambda : \ 0 \le \lambda < \infty\}$ is complete in $L^2[1, \infty]$. *You do not need to know anything about the conical functions beyond what you have just read.*

a) Write down the corresponding completeness relation. [10 points].

b) The *Mehler transform* $F(\lambda)$ of a function $f(x)$ defined on $[1, \infty]$ is given by

$$F(\lambda) \stackrel{\text{def}}{=} \int_1^\infty \varphi_\lambda(x) f(x)\,dx.$$

Use your completeness relation to write down the formula for the *inverse Mehler transform* that expresses $f(x)$ in terms of $F(\lambda)$. [10 points].

To receive credit in parts (a) and (b), you must have the correct limits on any integrals or sums that you write.