Physics 508 **Mathematical Methods in Physics I** Prof. M. Stone
Handout MT `https://courses.physics.illinois.edu/phys508/fa2015/` 2117 ESB
Nov 2nd 2015 **Midterm Exam** University of Illinois

Do all **three** *problems. Check each step before proceeding as there will be no propagation of errors. Marks will be subtracted for any equation that is obvious nonsense.*

1) Green Function: Consider the homogeneous boundary value problem

$$-y'' + m^2 y = f(x), \quad y(a) = y(b) = 0. \qquad (\star)$$

Here m^2 is a positive constant.

a) Find suitable solutions $y_L(x)$ and $y_R(x)$ that can be used to construct a Green function for this problem. [4 points]
b) Compute the Wronskian of your y_L and y_R. Verify that your answer is compatible with the Weierstrass formula applied to ($\star$). [4 points]
c) Construct the explicit Green function appropriate to this problem. [4 points]
d) Use your Green function to write down the solution of the boundary value problem as the sum of two explicit integrals over complementary components of the unit interval. [4 points]
e) Confirm that your solution $y(x)$ obeys both boundary conditions, and that it does indeed solve the original problem. [4 points]

2) Orthogonality and Completeness: The Macdonald functions $\mathrm{K}_\lambda(x)$ with purely imaginary index $\lambda = i\mu$ are real-valued when $0 < x < \infty$, and obey $\mathrm{K}_{i\nu}(x) = \mathrm{K}_{-i\nu}(x)$. They also possess the orthogonality property

$$\frac{1}{\pi^2}\int_0^\infty \frac{dx}{x}\mathrm{K}_{i\mu}(x)\mathrm{K}_{i\nu}(x) = \frac{\delta(\mu-\nu)}{2\nu\sinh\nu\pi}.$$

(You do not need to know anything about Macdonald functions other than what you have just been told!)

a) Assuming that these functions form a complete set for expanding out functions on $x > 0$, write down the *completeness relation* that expresses this fact. [10 points]
b) Given a function $f(x)$ defined for $x > 0$, we form its *Kontorovich-Lebedev* transform $\tilde{f}(\nu)$ by
$$\tilde{f}(\nu) = \int_0^\infty \mathrm{K}_{i\nu}(x) f(x)\, dx.$$
Write down the expression for the inverse transform that allows us to recover $f(x)$ from $\tilde{f}(\nu)$. [10 points]

3) Noether's theorem: Recall (you do **not** need to prove this) that a translation-invariant action integral

$$S[\varphi_a] = \int \mathcal{L}(\varphi_a, (\varphi_a)_\nu)\, d^d x, \quad \text{where} \quad (\varphi_a)_\mu \equiv \frac{\partial \varphi_a}{\partial x^\mu}$$

gives rise to a conserved *canonical energy-momentum tensor*

$$T^\mu{}_\nu = \sum_a \frac{\partial \mathcal{L}}{\partial (\varphi_a)_\mu} \partial_\nu \varphi_a - \mathcal{L} \delta^\mu_\nu.$$

a) Use this general formula with the assignment $\phi = \varphi_1$, $\rho = \varphi_2$ to find the energy current $T^\mu{}_0$, and the three components ($i = 1, 2, 3$) of the momentum current $T^\mu{}_i$, for the case that S is the action

$$S[\phi, \rho] = -\int dt\, d^3x \left\{ \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho (\nabla \phi)^2 + u(\rho) \right\}$$

for a barotropic fluid. [10 points]

b) Recall that the fluid velocity is given by $\mathbf{v} = \nabla \phi$ and show that the energy-momentum conservation law

$$\partial_\mu T^\mu{}_\nu = 0$$

leads to both the momentum conservation equation

$$\partial_t \{\rho v_i\} + \partial_j \{\rho v_i v_j + \delta_{ij} P\} = 0,$$

and to the energy conservation equation

$$\partial_t \mathcal{E} + \partial_i \{v_i (\mathcal{E} + P)\} = 0.$$

In this last equation you should have an explicit expression for the energy density $\mathcal{E}$ in terms of ρ, ϕ *etc.* [8 points]

c) Explain physically why both $\mathcal{E}$ and P appear in the energy current. [2 points]

Hint: A useful formula for the pressure is

$$P = -\left(\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho (\nabla \phi)^2 + u(\rho) \right).$$

— *End* —