

1) Linear differential operators:

- a) Let $w(x) > 0$. Consider the differential operator $\hat{L} = id/dx$. Find the formal adjoint of L with respect to the inner product $\langle u|v \rangle_w = \int wu^*v dx$, and find the corresponding surface term $Q[u, v]$.
- b) Now do the same for the operator $M = d^4/dx^4$, for the case $w = 1$. Find the adjoint boundary conditions defining the domain of M^\dagger for the case

$$\mathcal{D}(M) = \{y, y^{(4)} \in L^2[0, 1] : y(0) = y'''(0) = y(1) = y'''(1) = 0\}.$$

(Hint: you may find the identity

$$f^{(4)}g - fg^{(4)} = \frac{d}{dx} \{f'''g - f''g' + f'g'' - fg'''\}$$

to be of use.)

2) Sturm-Liouville forms: By constructing appropriate weight functions convert the following common operators into Sturm-Liouville form:

- a) $\hat{L} = (1 - x^2) d^2/dx^2 + [(\mu - \nu) - (\mu + \nu + 2)x] d/dx$.
- b) $\hat{L} = (1 - x^2) d^2/dx^2 - 3x d/dx$.
- c) $\hat{L} = d^2/dx^2 - 2x(1 - x^2)^{-1} d/dx - m^2(1 - x^2)^{-1}$.

3) Discrete approximations and self-adjointness: Consider the second order inhomogeneous equation $Lu \equiv u'' = g(x)$ on the interval $0 \leq x \leq 1$. Here $g(x)$ is known and $u(x)$ is to be found. We wish to solve the problem on a computer, and so set up a discrete approximation to the ODE in the following way:

- replace the continuum of independent variables $0 \leq x \leq 1$ by the discrete lattice of points $0 \leq x_n \equiv n/N \leq 1$ Here N is a positive integer and $n = 0, 1, 2, \dots, N$;
- replace the functions $u(x)$ and $g(x)$ by the arrays of real variables $u_n \equiv u(x_n)$ and $g_n \equiv g(x_n)$;
- approximate the continuum differential operator d^2/dx^2 by the finite difference operator \mathcal{D}^2 , defined by $\mathcal{D}^2 u_n \equiv (u_{n+1} - 2u_n + u_{n-1})/a^2$ where $a = N^{-1}$ is the lattice spacing.

Now do the following problems:

- a) Impose continuum Dirichlet boundary conditions $u(0) = u(1) = 0$. Decide what these correspond to in the discrete approximation, and write the resulting set of algebraic equations in matrix form. Show that the corresponding matrix is real and symmetric.
- b) Impose the periodic boundary conditions $u(0) = u(1)$ and $u'(0) = u'(1)$, and show that these require us to set $u_0 \equiv u_N$ and $u_{N+1} \equiv u_1$. Again write the system of algebraic equations in matrix form and show that the resulting matrix is real and symmetric.

c) Consider the non-symmetric $N \times N$ matrix operator

$$D^2 u = \frac{1}{a^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_N \\ u_{N-1} \\ u_{N-2} \\ \vdots \\ u_3 \\ u_2 \\ u_1 \end{pmatrix}.$$

- i) What vectors span the null space of D^2 ?
- ii) To what continuum boundary conditions for d^2/dx^2 does this matrix correspond?
- iii) Consider the matrix $(D^2)^\dagger$, To what continuum boundary conditions does this matrix correspond? Are they the adjoint boundary conditions for the operator in part ii)?

4) Factorization: Schrödinger equations of the form

$$-\frac{d^2\psi}{dx^2} - l(l+1)\operatorname{sech}^2 x \psi = E\psi$$

are known as *Pöschel-Teller equations*. By setting $u = l \tanh x$ and following the strategy of this problem one may relate solutions for l to those for $l - 1$ and so find all bound states and scattering eigenfunctions for any integer l .

a) Suppose that we know that $\psi = \exp\{-\int^x u(x')dx'\}$ is a solution of

$$L\psi \equiv \left(-\frac{d^2}{dx^2} + W(x)\right)\psi = 0.$$

Show that L can be written as $L = M^\dagger M$ where

$$M = \left(\frac{d}{dx} + u(x)\right), \quad M^\dagger = \left(-\frac{d}{dx} + u(x)\right),$$

the adjoint being taken with respect to the product $\langle u|v \rangle = \int u^* v dx$.

b) Now assume L is acting on functions on $[-\infty, \infty]$ and that we not have to worry about boundary conditions. Show that given an eigenfunction ψ_- obeying $M^\dagger M\psi_- = \lambda\psi_-$ we can multiply this equation on the left by M and so find a eigenfunction ψ_+ with the same eigenvalue for the differential operator

$$L' = MM^\dagger = \left(\frac{d}{dx} + u(x)\right) \left(-\frac{d}{dx} + u(x)\right)$$

and *vice-versa*. Show that this correspondence $\psi_- \leftrightarrow \psi_+$ will fail if, *and only if*, $\lambda = 0$.

- c) Apply the strategy from part b) in the case $u(x) = \tanh x$ and one of the two differential operators $M^\dagger M$, MM^\dagger is (up to an additive constant)

$$H = -\frac{d^2}{dx^2} - 2 \operatorname{sech}^2 x.$$

Show that H has eigenfunctions of the form $\psi_k = e^{ikx}P(\tanh x)$ and eigenvalue $E = k^2$ for any k in the range $-\infty < k < \infty$. The function $P(\tanh x)$ is a polynomial in $\tanh x$ which you should be able to find explicitly. By thinking about the exceptional case $\lambda = 0$, show that H has an eigenfunction $\psi_0(x)$, with eigenvalue $E = -1$, that tends rapidly to zero as $x \rightarrow \pm\infty$. Observe that there is no corresponding eigenfunction for the other operator of the pair.