Physics 508/MMA **Mathematical Methods in Physics I** Prof. M. Stone
Handout MT `http://webusers.physics.uiuc.edu/~m-stone5` 2117 ESB
Oct 24tht 2011 MIDTERM EXAM University of Illinois

Do all the questions. Little, if any, credit will be given for fragmentary answers.

1) Green Function: Consider the *homogeneous* boundary value problem

$$-y'' = f(x), \quad y'(0) = y(1) = 0.$$

a) Construct the explicit Green function appropriate to this problem. (3 points)
b) Use your Green function to write down the solution of the boundary value problem as the sum of two explicit integrals over complementary components of the unit interval. (3 points)
c) Confirm that your solution $y(x)$ obeys both boundary conditions, and that it does indeed solve the original problem. (4 points)

Now consider the *inhomogeneous* boundary value problem

$$-y'' = f(x), \quad y'(0) = A, \quad y(1) = B.$$

d) Use the method based Lagrange's identity to obtain the solution to this boundary value problem. The points here are for exhibiting the *method*, so merely writing down the solution will not earn any credit. (10 points)

2) Orthogonality and Completeness: The functions $\psi_n(x) = H_n(x)e^{-\frac{1}{2}x^2}$ obey

$$\left(-\frac{d^2}{dx^2} + x^2\right)\psi_n(x) = (2n+1)\psi_n(x), \quad x \in \mathbf{R}.$$

Here $H_n(x)$, $n = 0, 1, 2, \ldots$ are the Hermite polynomials.

a) Using *only the information given in this problem*, show that

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = 0, \quad n \neq m.$$

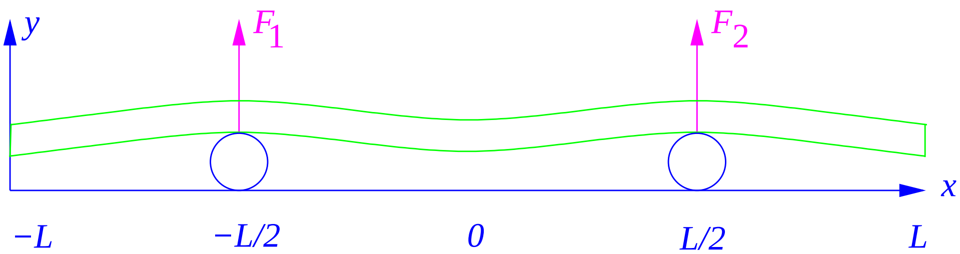
(10 points)

b) Given that

$$\int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}\,dx = 2^n n!\sqrt{\pi},$$

write down the expected *completeness relation* for the eigenfunctions in terms of the ψ_n. (10 points)

3) Standard Meter : The original meter was defined by the distance between two marks engraved into the top surface of a platinum bar. The bar was to be supported on two rollers as shown in the figure:



The standard meter and its supports.

The bar tended to sag.

The potential energy of the bar is given by

$$E(y) = \int_{-L}^{L} \left\{ \frac{\kappa}{2}(y'')^2 + \rho g y - F_1 y\, \delta(x + L/2) - F_2 y\, \delta(x - L/2) \right\} dx,$$

where κ is an elastic constant, and ρg the weight of the bar per unit length. By using the calculus of variations:

a) Derive the differential equation whose solution is the equilibrium shape $y(x)$ of the sagging bar. (7 points)

b) Derive the boundary conditions obeyed by $y(x)$ at the endpoints $x = \pm L$. (7 points)

c) Explain, by invoking the *Fredholm alternative*, how the existence of a solution of this equation allows us to determine the forces $F_{1,2}$ exerted by the supports. (6 points)

— *End* —