1) Critical Mass: An infinite slab of fissile material has thickness L. The neutron density $n(\mathbf{r})$ in the material obeys the equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + \lambda n + \mu,$$

where n is zero at the surface of the slab at $x = 0, L$. Here D is the neutron diffusion constant, the term λn describes the creation of new neutrons by induced fission, and μ is the rate of production per unit volume of neutrons by spontaneous fission. Assume that n depends only on x and t, and that λ and μ are constants,

a) Expand both n and μ as series

$$n(x,t) = \sum_m a_m(t)\varphi_m(x), \quad \mu = \sum_m b_m \varphi_m(x)$$

where the φ_m are a complete orthonormal set of functions you think suitable for solving the problem.

b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their intial values $a_m(0)$.

c) Determine the critical thickness, L_{crit}, above which the slab will explode.

d) Assuming that $L < L_{\text{crit}}$, find the equilibrium distribution $n_{\text{eq}}(x)$ of neutrons in the slab. (You may either sum your series expansion to get an explicit closed-form answer, or use another (Green function?) method.)

2) Semi-infinite Rod: Consider the heat equation

$$\frac{\partial \theta}{\partial t} = D\nabla^2\theta, \quad 0 < x < \infty$$

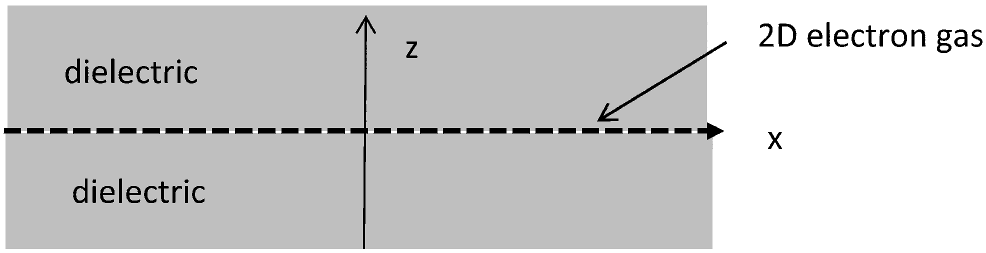
with the temperature $\theta(x,t)$ obeying the initial condition $\theta(x,0) = \theta_0$ for $0 < x < \infty$, and the boundary condition $\theta(0,t) = 0$.

a) Show that the boundary condition at $x = 0$ can be satisfied at all times by introducing a suitable mirror image of the initial data in the region $-\infty < x < 0$, and then applying the heat kernel for the entire real line to this extended initial data. Show that the solution of the semi-infinite rod problem can be expressed in terms of the *error function*

$$\text{erf}\, x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.$$

b) Solve the same problem by using a Fourier integral expansion in terms of $\sin kx$ on the half-line $0 < x < \infty$ and obtaining the time evolution of the Fourier coefficients. Invert the transform and show that your answer reduces to that of part a). (Hint: replace the initial condition by $\theta(x,0) = \theta_0 e^{-\epsilon x}$, so that the Fourier transform converges, and then take the limit $\epsilon \to 0$ at the end of your calculation.)

3) 2-D Electron Gas — an old Qual problem:



A two-dimensional gas of electrons is confined at the $z = 0$ interface between two semi-infinite dielectric slabs. Each slab has dielectric constant ε. A perturbation of the electron charge-density propagates as a wave through the electron gas. The surface-charge density on the interface is therefore given by $\sigma(x,t) = \sigma_0 + \sigma_1(x,t)$, where σ_0 is constant and the small-amplitude perturbation σ_1 takes the form

$$\sigma_1(x,t) = a \exp\{i(kx - \omega t)\}.$$

Assume that electrons act as classical particles of mass m with local velocity,

$$v(x,t) = v_0 \exp\{i(kx - \omega t)\},$$

and that the only significant force is due to the electric field produced by the charge density perturbation.

a) Use Laplace's equation

$$-\nabla^2 \phi = \varepsilon^{-1} \delta(z) \sigma(x,t)$$

to find the electrical potential $\phi(x,z,t)$ due to the charge.

b) From $\phi(x,z,t)$ find the electric field component $E_x(x, z = 0, t)$ parallel to and within the electron gas, and hence the acceleration $\partial v(x,t)/\partial t$ of the electrons.

c) Linearize the charge continuity equation

$$\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma v}{\partial x} = 0,$$

and use it to relate a and v_0. Hence show that the dispersion equation relating the frequency ω to the wavenumber k is

$$\omega^2 = \gamma |k|.$$

Express the coefficient γ in terms of m, ε, σ_0 and the electron charge $q = -e$.

4) Seasonal Heat Waves: Suppose that the measured temperature of the air above the arctic permafrost is expressed as a Fourier series

$$\theta(t) = \theta_0 + \sum_{n=1}^{\infty} \theta_n \cos n\omega t,$$

where $T = 2\pi/\omega$ is one year. Solve the heat equation for the soil temperature

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}, \quad 0 < z < \infty$$

with this boundary condition, and find the temperature $\theta(z, t)$ at a depth z below the surface as a function of time. Observe that the sub-surface temperature fluctuates with the same period as that of the air, but with a phase lag that depends on the depth. Also observe that the longest period temperature fluctuations penetrate the deepest into the ground. (Hint: for each Fourier component, write θ as $\mathrm{Re}[A_n(z) \exp in\omega t]$ where A_n is a complex function of z.)