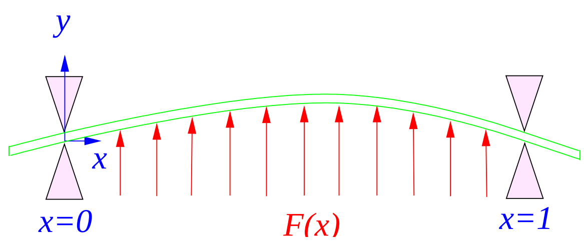
Physics 508 | Prof. M. Stone
Handout 7 | **Mathematical Methods in Physics I** | 2117 ESB
Fall 2023 | HOMEWORK 7 | University of Illinois

1) Flexible rod again: A flexible rod is supported near its ends by means of knife edges that constrain its position, but not its slope or curvature. It is acted on by by a force $F(x)$.



Simply supported rod.

The deflection of the rod is found by solving the the boundary value problem

$$\frac{d^4y}{dx^4} = F(x), \quad y(0) = y(1) = 0, \quad y''(0) = y''(1) = 0.$$

We wish to find the Green function $G(x, y)$ that facilitates the solution of this problem.

a) If the differential operator and domain (boundary conditions) above is denoted by L, what is the operator and domain for $L^\dagger$? Is the problem self-adjoint?

b) Are there any zero-modes? Does F have to satisfy any conditions for the solution to exist?

c) Write down the conditions, if any, obeyed by $G(x, y)$ and its derivatives $\partial_x G(x, y)$, $\partial^2_{xx} G(x, y)$, $\partial^3_{xxx} G(x, y)$ at $x = 0$, $x = y$, and $x = 1$.

d) Using the conditions above, find $G(x, y)$. (This requires some boring algebra — but if you start from the "jump condition" and work down, it can be completed in under a page)

e) Is your Green function symmetric ($G(x, y) = G(y, x)$)? Is this in accord with the self-adjointness or not of the problem? (You can use this as a check of your algebra.)

f) Write down the integral giving the general solution of the boundary value problem. Assume, if necessary, that $F(x)$ is in the range of the differential operator. Differentiate your answer and see if it does indeed satisfy the differential equation and boundary conditions.

2) Hot ring: The equation governing the steady state heat flow on thin ring of unit circumference is

$$-\varphi'' = f, \quad 0 < x < 1, \quad \varphi(0) = \varphi(1), \quad \varphi'(0) = \varphi'(1).$$

a) This problem has a zero mode. Find the zero mode and the consequent condition on $f(x)$ for a solution to exist.

b) Verify that a suitable modified Green function for the problem is

$$g(x,y) = \frac{1}{2}(x-y)^2 - \frac{1}{2}|x-y|.$$

You will need to verify that $g(x,y)$ satisfies both the differential equation *and* the boundary conditions.

3) Lattice Green Functions . The $k \times k$ matrices

$$\mathbf{T}_1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

represent two discrete lattice approximations to $-\partial_x^2$ on a finite interval.

a) What are the boundary conditions defining the domains of the corresponding continuum differential operators? [They are either Dirichlet ($y = 0$) or Neumann ($y' = 0$) boundary conditions.] Make sure you explain your reasoning.

b) Verify that

$$\begin{aligned} [\mathbf{T}_1^{-1}]_{ij} &= \min(i,j) - \frac{ij}{k+1}, \\ [\mathbf{T}_2^{-1}]_{ij} &= \min(i,j). \end{aligned}$$

c) Find the continuum Green functions for the boundary value problems approximated by the matrix operators. Compare each of the matrix inverses with its corresponding continuum Green function. Are they similar?

4) Eigenfunction expansion : We saw in class that the resolvent (Green function) $G_\lambda(x,x') = (L-\lambda)^{-1}_{xx'}$ can be expanded as

$$(L-\lambda)^{-1}_{xx'} = \sum_{\lambda_n} \frac{\varphi_n(x)\varphi_n(x')}{\lambda_n - \lambda},$$

where $\varphi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue λ_n. The resolvent therefore has a *pole* whenever λ approaches λ_n. Consider the case

$$G_{\omega^2}(x,x') = \left(-\frac{d^2}{dx^2} - \omega^2\right)^{-1}_{xx'},$$

with boundary conditions $y(0) = y(L) = 0$. By using the explicit expression for $G_{\omega^2}(x,x')$ that is given in the first problem in Homework set 1:

a) Confirm that G_{ω^2} becomes singular at exactly those values of ω^2 corresponding to eigenvalues of $-\frac{d^2}{dx^2}$.
b) Confirm that the *residue* of the pole (the coefficient of $1/(\omega_n^2 - \omega^2)$) is precisely the product of the *normalized* eigenfunctions $\varphi_n(x)\varphi_n(x')$.