Physics 508		Prof. M. Stone
Handout 11	**Mathematical Methods in Physics I**	2117 ESB
Fall 2023	HOMEWORK 11	University of Illinois

1) Dielectric Sphere: Consider a solid dielectric sphere of radius a and permittivity ϵ. The sphere is placed in a electric field which is takes the constant value $\mathbf{E} = E_0\hat{\mathbf{z}}$ a long distance from the sphere. Recall that Maxwell's equations require that $\mathbf{D}_\perp$ and $\mathbf{E}_\parallel$ be continuous across the surface of the sphere.

a) Use the expansions

$$\begin{aligned}\Phi_{in} &= \sum_l A_l r^l P_l(\cos\theta)\\ \Phi_{out} &= \sum_l (B_l r^l + C_l r^{-l-1}) P_l(\cos\theta)\end{aligned}$$

and find all non-zero coefficents A_l, B_l, C_l.

b) Show that the $\mathbf{E}$ field inside the sphere is uniform and of magnitude $\frac{3\epsilon_0}{\epsilon+2\epsilon_0}E_0$.

c) Show that the electric field is unchanged if the dielectric is replaced by the polarization-induced surface charge density

$$\sigma_{\text{induced}} = 3\epsilon_0\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)E_0\cos\theta.$$

(Some systems of units may require extra 4π's in this last expression. In SI units $\mathbf{D} \equiv \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$, and the polarization induced charge density is $\rho_{\text{induced}} = -\nabla\cdot\mathbf{P}$)

2) Hollow Sphere: The potential on a spherical surface of radius a is $\Phi(\theta, \phi)$. We want to express the potential inside the sphere as an integral over the surface in a manner analagous to the Poisson kernel in two dimensions.

a) By using the generating function for Legendre polynomials, show that

$$\frac{1-r^2}{(1+r^2-2r\cos\theta)^{3/2}} = \sum_{l=0}^{\infty}(2l+1)r^l P_l(\cos\theta), \quad r<1$$

b) Starting from the expansion

$$\begin{aligned}\Phi_{in}(r,\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} r^l Y_m^l(\theta,\phi)\\ A_{lm} &= \frac{1}{a^l}\int_{S^2}\left[Y_m^l(\theta,\phi)\right]^*\Phi(\theta,\phi)\,d\cos\theta\,d\phi\end{aligned}$$

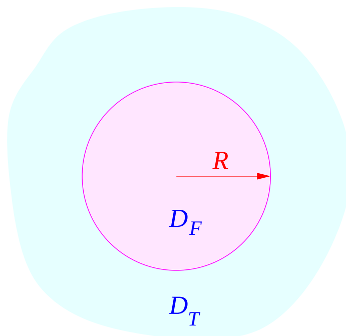
and using the addition formula for spherical harmonics, show that

$$\Phi_{in}(r,\theta,\phi) = \frac{a(a^2-r^2)}{4\pi}\int_{S^2}\frac{\Phi(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}d\cos\theta' d\phi'$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

c) By setting $r = 0$, deduce that a three dimensional harmonic function cannot have a local maximum or minimum.

3) Peierls Problem: Computing the Critical Mass. The core of a fast breeder reactor consists of a sphere of fissile ^{235}U of radius R. It is surrounded by a thick shell of non-fissile material which acts as a neutron reflector, or *tamper*.



Fast breeder reactor.

In the core, the fast neutron density $n(\mathbf{r}, t)$ obeys

$$\frac{\partial n}{\partial t} = \nu\, n + D_F \nabla^2 n.$$

Here the term with ν accounts for the production of additional neutrons due to induced fission. The term with D_F describes the diffusion of the fast neutrons. In the tamper the neutron flux obeys

$$\frac{\partial n}{\partial t} = D_T \nabla^2 n.$$

Both the neutron density, n, and the neutron flux, $\mathbf{j} = D_{F,T}\nabla n$, are continuous across the interface between the two materials. Find an equation determining the critical radius, R_c, above which the neutron density grows without bound. Show that the critical radius for an assembly with a tamper consisting of ^{238}U ($D_T = D_F$) is one-half of that for a core surrounded only by air ($D_T = \infty$), and so the use of a thick ^{238}U tamper reduces the critical mass by a factor of eight.

4) Bessel functions and impact parameters: In two dimensions we can expand a plane wave as

$$e^{iky} = \sum_{n=-\infty}^{\infty} J_n(kr) e^{in\theta}.$$

a) What do you think the resultant wave will look like if we take only a finite segment of this sum? For example

$$\phi(x) = \sum_{l=10}^{17} J_n(kr) e^{in\theta}.$$

Think about:

i) The quantum interpretation of $\hbar l$ as angular momentum $= \hbar k d$, where d is the *impact parameter*, the amount by which the incoming particle misses the origin.

ii) Diffraction: one cannot have a plane wave of finite width.

b) After (I trust you!) writing down your best guess for the previous part, confirm your understanding by using Mathematica or other package to plot the real part of ϕ as defined above. The following Mathematica code may work.

```
Clear[bit,tot]
bit[l_,x_,y_]:=Cos[l ArcTan[x,y]]BesselJ[l,Sqrt[x^2+y^2]]
tot[x_,y_] :=Sum[bit[l,x,y],{l,10,17}]
ContourPlot[tot[x,y],{x,-40,40},{y,-40,40},PlotPoints ->200]
Display["wave",%,"EPS"]
```

Try different ranges for the sum. Include a printout of your code together with the plot.

5) Berry's problem: Levitating Frogs (Optional). Sir Michael Berry (of *Berry's phase* fame) shared the 2000 IgNobel Prize for his work establishing the theoretical basis for the diamagnetic levitation of frogs. The picture shows a live frog floating happily in the bore of a 16 T Bitter Solenoid at the High Field Magnet Lab in the University of Nijmegen. The experiment was performed by Andre Geim, who shared the IgNobel with with Berry and the won the real Nobel prize in 2010 for his work on graphene — another strongly diamagnetic material. (A movie of the floating frog is at https://www.youtube.com/watch?v=A1vyB-O5i6E).

Levitated frog.

a) *Earnshaw's theorem*: Show that an object that experiences the combined force of gravity and a magnetic force $\propto \mathbf{B}$, cannot have a position of stable equilibrium.

b) Show that, in a current-free region, the magnitude of the magnetic field obeys

$$\nabla^2|\mathbf{B}|^2 = 2(|\nabla B_x|^2 + |\nabla B_y|^2 + |\nabla B_z|^2) \geq 0.$$

c) Deduce that $|\mathbf{B}|^2$ can never have a local *maximum*, but may have a local *minimum*.

d) Consider the field in the bore of a cylindrical solenoid. Write $\mathbf{B} = \nabla\phi$, where ϕ is the magnetic potential, and define $\phi_n(z) = \partial_z^n \phi(0,0,z)$, the derivatives along the bore axis. Use cylindrical symmetry, together with the fact that ϕ is harmonic, to show that near the bore axis

$$\phi(x,y,z) \approx \phi_0(z) - \frac{1}{4}(x^2+y^2)\phi_2(z),$$

and hence

$$|\mathbf{B}|^2 \approx \phi_1^2(z) + \frac{1}{4}(x^2+y^2)(\phi_2^2(z) - 2\phi_1(z)\phi_3(z)).$$

Deduce (Berry and Gaim) that the second derivatives of $|\mathbf{B}|^2$ are all positive when

$$\begin{aligned} D_1(z) &\equiv B'(z)^2 + B(z)B''(z) > 0, \quad \text{vertical stability,} \\ D_2(z) &\equiv B'(z)^2 - 2B(z)B''(z) > 0, \quad \text{horizontal stability.} \end{aligned}$$

Here $B(z) = |\mathbf{B}(0,0,z)|$ is the magnitude of the field on the bore axis. Observe that both these inequalities will be satisfied in some vicinity of a point of inflection of $B(z)$.

e) Deduce that a *paramagnetic* object, which is attracted to regions of strong $|\mathbf{B}|^2$, can never be stably levitated, but that a *diamagnetic* frog — which is predominantly water — may be stably levitated.