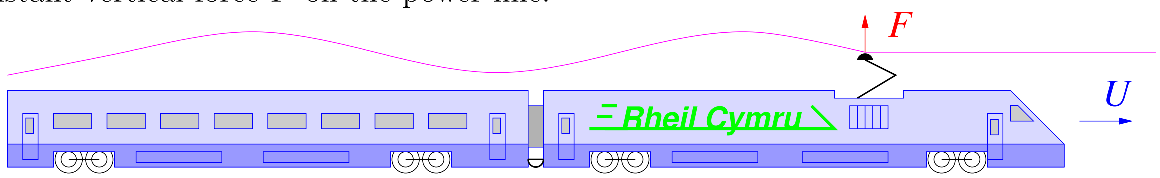
Mathematical Methods in Physics I

Homework 9

1) Pantograph Drag: A high-speed train picks up its electrical power via a pantograph from an overhead line. The locomotive travels at speed U and the pantograph exerts a constant vertical force F on the power line.



A high-speed train.

We make the usual small amplitude approximations and assume (not unrealistically) that the line is supported in such a way that its vertical displacement obeys an inhomogeneous Klein-Gordon equation

$$\rho\ddot{y} - Ty'' + \rho\Omega^2 y = F\delta(x - Ut),$$

with $c = \sqrt{T/\rho}$ being the velocity of propagation of short-wavelength transverse waves on the overhead cable.

a) Assume that $U < c$ and solve for the steady state displacement of the cable about the pickup point. (Hint: the disturbance is time-independent when viewed from the train.)

b) Now assume that $U > c$. Again find an expression for the displacement of the cable. (The same hint applies, but the physically appropriate boundary conditions are very different!)

c) By equating the rate at which wave-energy

$$E = \int \left\{ \frac{1}{2}\rho\dot{y}^2 + \frac{1}{2}Ty'^2 + \frac{1}{2}\rho\Omega^2 y^2 \right\} dx$$

is being created to rate at the which the locomotive is doing work, calculate the wave-drag on the train. In particular, show that there is no drag at all until U exceeds c. (Hint: While the front end of the wake is moving at speed U, the trailing end of the wake is moving forward at the *group velocity* of the wave-train.)

This problem of wake formation and drag is related both to Čerenkov radiation and to the Landau criterion for superfluidity.

2) Non-linear Waves:

a) Suppose a fluid has equation of state $P = \lambda^2\rho^3/3$. From the continuity equation

$$\partial_t\rho + \partial_x \rho v = 0,$$

and Euler's equation of motion

$$\rho(\partial_t v + v\partial_x v) = -\partial_x P,$$

deduce that

$$\begin{aligned}\left(\frac{\partial}{\partial t} + (\lambda\rho + v)\frac{\partial}{\partial x}\right)(\lambda\rho + v) &= 0,\\ \left(\frac{\partial}{\partial t} + (-\lambda\rho + v)\frac{\partial}{\partial x}\right)(-\lambda\rho + v) &= 0.\end{aligned}$$

By considering the case of small perturbations around an equilibrium fluid which is at rest, show that these equations are equivalent to the ordinary one-dimensional wave equation. What is the sound speed in this case?

b) Show that the *Riemann invariants* $v \pm \lambda\rho$ are constant on suitably defined characteristic curves. What is the local speed of propagation of the waves moving to the right or left?

c) The fluid starts from rest, $v = 0$, but with a region where the density is higher than elsewhere. Show that that the Riemann equations will inevitably break down at some later time due to the formation of shock waves.

The notion of Riemann invariants can be extended to other equations of state, but the expressions are more complicated, and the left and right-going waves interact with each other.

3) Burgers Shocks: As simple mathematical model for the formation and decay of a shock wave consider *Burgers' Equation*:

$$\partial_t u + u\partial_x u = \nu\, \partial_x^2 u.$$

Note its similarity to the Riemann equations of the previous question. As discussed in class, if $\nu = 0$ any solution of Burgers' equation having a region where u decreases to the right will always eventually become multivalued. The ν term introduces dissipation and prevents the solution becoming multi-valued.

a) Show that the *Hopf-Cole* transformation, $u = -2\nu\, \partial_x \ln\psi$, leads to ψ obeying a heat diffusion equation

$$\partial_t \psi = \nu\, \partial_x^2 \psi.$$

b) Show that

$$\psi(x,t) = Ae^{\nu a^2 t - ax} + Be^{\nu b^2 t - bx}$$

is a solution of the heat equation, and so deduce that Burgers' equation has a shock-wave-like solution which travels to the right at speed $C = \nu(a+b) = \frac{1}{2}(u_L + u_R)$, the mean of the wave speeds to the left and right of the shock. Show that the width of the shock is $\approx 4\nu/|u_L - u_R|$.