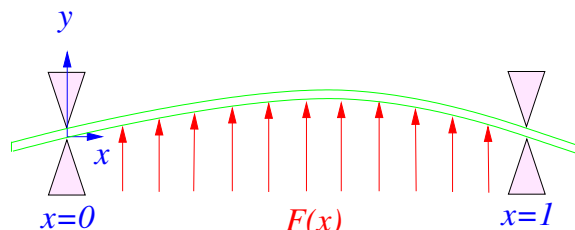


# Mathematical Methods in Physics I

## HOMEWORK 7

**1) Flexible rod again:** A flexible rod is supported near its ends by means of knife edges that constrain its position, but not its slope or curvature. It is acted on by a force  $F(x)$ .



*Simply supported rod.*

The deflection of the rod is found by solving the the boundary value problem

$$\frac{d^4 y}{dx^4} = F(x), \quad y(0) = y(1) = 0, \quad y''(0) = y''(1) = 0.$$

We will construct the Green function  $G(x, y)$  that solves this problem.

- If the differential operator and domain (boundary conditions) above is denoted by  $L$ , what is the operator and domain for  $L^\dagger$ ? Is the problem self-adjoint?
- Are there any zero-modes? Does  $F$  have to satisfy any conditions for the solution to exist?
- Write down the conditions, if any, obeyed by  $G(x, y)$  and its derivatives  $\partial_x G(x, y)$ ,  $\partial_{xx}^2 G(x, y)$ ,  $\partial_{xxx}^3 G(x, y)$  at  $x = 0$ ,  $x = y$ , and  $x = 1$ .
- Using the conditions above, find  $G(x, y)$ . (This requires some boring algebra — but if you start from the “jump condition” and work down, it can be completed in under a page)
- Is your Green function symmetric ( $G(x, y) = G(y, x)$ )? Is this in accord with the self-adjointness or not of the problem? (You can use this as a check of your algebra.)
- Write down the integral giving the general solution of the boundary value problem. Assume, if necessary, that  $F(x)$  is in the range of the differential operator. Differentiate your answer and see if it does indeed satisfy the differential equation and boundary conditions.

**2) Hot ring:** The equation governing the steady state heat flow on thin ring of unit circumference is

$$-y'' = f, \quad 0 < x < 1, \quad y(0) = y(1), \quad y'(0) = y'(1).$$

- a) This problem has a zero mode. Find the zero mode and the consequent condition on  $f(x)$  for a solution to exist.
- b) Verify that a suitable modified Green function for the problem is

$$g(x, y) = \frac{1}{2}(x - y)^2 - \frac{1}{2}|x - y|.$$

You will need to verify that  $g(x, y)$  satisfies both the differential equation *and* the boundary conditions.

**3) Lattice Green Functions .** The  $k \times k$  matrices

$$\mathbf{T}_1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

represent two discrete lattice approximations to  $-\partial_x^2$  on a finite interval.

- a) What are the boundary conditions defining the domains of the corresponding continuum differential operators? [They are either Dirichlet ( $y = 0$ ) or Neumann ( $y' = 0$ ) boundary conditions.] Make sure you explain your reasoning.
- b) Verify that

$$\begin{aligned} [\mathbf{T}_1^{-1}]_{ij} &= \min(i, j) - \frac{ij}{k+1}, \\ [\mathbf{T}_2^{-1}]_{ij} &= \min(i, j). \end{aligned}$$

- c) Find the continuum Green functions for the boundary value problems approximated by the matrix operators. Compare each of the matrix inverses with its corresponding continuum Green function. Are they similar?