Physics 508 | Mathematical Methods in Physics I | Prof. P. Draper
Homework 4 | | 435 Loomis
Fall 2021 | HOMEWORK 4 | University of Illinois

*****Due Fri 9/24 5pm*****

1) Test functions and distributions:

a) Let $f(x)$ be a smooth function.

i) Show that $f(x)\delta(x) = f(0)\delta(x)$. Therefore we can conclude that $\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x)$.

ii) We might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x).$$

By integrating both against a test function, show this new expression for the derivative of $f(x)\delta(x)$ is equivalent to that in part i).

b) In an old paper[1] that has been cited in the literature on topological insulators a distribution $\delta^{(1/2)}(x)$ is defined by setting

$$\delta^{(1/2)}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^{1/2} e^{ikx}.$$

The Fourier transform on the RHS is clearly divergent, so we need to decide how to interpret it. Let's try to define the evaluation of $\delta^{(1/2)}$ on a test function $\varphi(x)$ as

$$\int_{-\infty}^{\infty} \delta^{(1/2)}(x)\varphi(x)\, dx \stackrel{\text{def}}{=} \lim_{\mu\to 0^+} \left\{ \int_{-\infty}^{\infty} \delta_\mu^{(1/2)}(x)\varphi(x)\, dx \right\}. \tag{1}$$

where

$$\begin{aligned}\delta_\mu^{(1/2)}(x) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{ikx}|k|^{1/2}e^{-\mu|k|}\,\frac{dk}{2\pi} \\ &= \sqrt{\frac{1}{4\pi}}(x^2+\mu^2)^{-3/4}\cos\left(\frac{3}{2}\tan^{-1}\left(\frac{x}{\mu}\right)\right).\end{aligned} \tag{2}$$

The extra factor of $e^{-\mu|k|}$ in the integrand is called a convergence factor which suppresses the integrand at large $|k|$ (for $\mu > 0$).[2]

To understand the regularized $\delta^{1/2}$ given by Eq. (2),

(i) Sketch plots of $\delta_\mu^{(1/2)}(x)$ for various values of μ, and so get an idea of how it behaves as the convergence factor $e^{-\mu|k|} \to 1$. Where are the zeros of Eq. (2)?

[1] H. Aratyn, *Fermions from Bosons in 2+1 dimensions*, Phys. Rev. D **28** (1983) 2016-18.

[2] To evaluate the integral, split the integration region into $k < 0$ and $k > 0$, change the variable to $y = |k|^{1/2}$, write the result as $d/d\mu$ of an ordinary Gaussian integral, evaluate the Gaussian integral and the μ derivative, and finally use the trigonometric relation $\arctan(z) = \frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right)$.

(ii) Observe that $\delta_\mu^{(1/2)}(x)$ is the Fourier transform of a function that vanishes at $k = 0$. What property of the graph of $\delta_\mu^{(1/2)}(x)$ does this imply?
(iii) By taking $\mu \to 0$ at fixed x in Eq. (2) above, obtain the leading behavior of $\delta_\mu^{(1/2)}(x \gg \mu)$.
Use (i) and (ii) to rewrite the integral on the right-hand side of Eq. (1) as an integral over the domain $|x|$ greater than the zeros of $\delta_\mu^{(1/2)}$. Since for any x away from the origin $\delta_\mu^{(1/2)}(x)$ converges pointwise to the leading behavior in (iii) in the limit $\mu \to 0^+$, deduce that Eq. (1) converges to the result

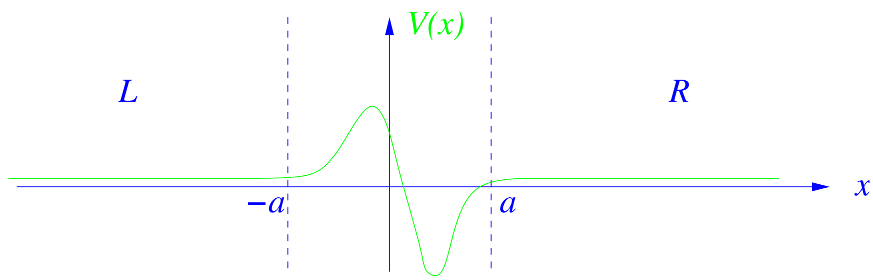
$$\int_{-\infty}^{\infty} \delta^{(1/2)}(x)\varphi(x)\,dx = -\sqrt{\frac{1}{8\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{3/2}} \{\varphi(x) - \varphi(0)\}\,dx.$$

This shows that $\delta^{(1/2)}(x)$ is indeed a sensible distribution with finite action on test functions.

2) One-dimensional scattering theory: Consider the one-dimensional Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \quad V(x) \in \mathbb{R},$$

where $V(x)$ is zero except in a finite interval $[-a, a]$ near the origin, e.g. something like:



Let L denote the left asymptotic region, $-\infty < x < -a$, and similarly let R denote $\infty > x > a$. For $E = k^2$ and $k > 0$ there will be scattering solutions of the form

$$\psi_k(x) = \begin{cases} e^{ikx} + r_L(k)e^{-ikx}, & x \in L \\ t_L(k)e^{ikx}, & x \in R \end{cases}$$

describing waves incident on the potential $V(x)$ from the left. For $k < 0$ there will be solutions with waves incident from the right

$$\psi_k(x) = \begin{cases} t_R(k)e^{ikx}, & x \in L \\ e^{ikx} + r_R(k)e^{-ikx}, & x \in R \end{cases}$$

The wavefunctions in $[-a, a]$ will naturally be more complicated. Observe that $[\psi_k(x)]^*$ is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:

a) $|r_{L,R}|^2 + |t_{L,R}|^2 = 1$,
b) $t_L(k) = t_R(-k)$.
c) Deduce from parts a) and b) that $|r_L(k)| = |r_R(-k)|$.
d) Take the specific example of $V(x) = \lambda\delta(x-b)$ with $|b| < a$. Compute the transmission and reflection coefficients and hence show that $r_L(k)$ and $r_R(-k)$ may differ by a phase.

3) Reduction of Order: Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that $y_1 = u(x)$ is one solution to the equation

$$y'' + V(x)y = 0.$$

By trying $y = u(x)v(x)$ show that

$$y_2 = u(x)\int^x \frac{d\xi}{u^2(\xi)}$$

is also a solution of the differential equation. Is this new solution ever merely a constant mutiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian $W(y_2, y_1)$.)

b) Suppose that we are told that the product, $y_1 y_2$, of the two solutions to the equation $y'' + p_1(x)y' + p_2(x)y = 0$ is a constant. Show that this requires $2p_1p_2 + p_2' = 0$.

c) By using ideas from part b) or otherwise, find the general solution of the equation

$$(x+1)x^2y'' + xy' - (x+1)^3y = 0.$$

4) Normal forms and the Schwarzian derivative: We saw in class that if y obeys a second-order linear differential equation

$$y'' + p_1y' + p_2y = 0$$

then we can make always make a substitution $y = w\tilde{y}$ so that $\tilde{y}$ obeys an equation without a first derivative:

$$\tilde{y}'' + q(x)\tilde{y} = 0.$$

Suppose $\psi(x)$ obeys a Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + [V(x) - E]\right)\psi = 0.$$

a) Make a smooth and invertible change of independent variable by setting $x = x(z)$ and find the second order differential equation in z obeyed by $\psi(z) \equiv \psi(x(z))$. This equation will have a term with a first derivative of ψ. Find the $\tilde{\psi}(z)$ that obeys an equation with no first derivative. Show that this equation is

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} + (x')^2[V(x(z)) - E] - \frac{1}{4}\{x, z\}\right)\tilde{\psi}(z) = 0,$$

where the primes denote differentiation with respect to z, and

$$\{x, z\} \equiv \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2$$

is called the *Schwarzian* derivative of x with respect to z. Schwarzian derivatives play an important role in conformal field theory and string theory.

b) Now combine a sequence of coordinate changes $x \to z \to q$. Establish *Cayley's identity*:

$$\left(\frac{dz}{dq}\right)^2 \{x, z\} + \{z, q\} = \{x, q\}.$$

(Hint: If this takes you more than a line or two, or you find yourself using the hideous expression for $\{x, z\}$, you are missing the point of the problem.)