Mathematical Methods in Physics I

HOMEWORK 10

1) Tides and Gravity : The Earth is not exactly spherical. Two major causes of the deviation from sphericity are the Earth's rotation and the tidal forces it feels from the Sun and the Moon. In this problem we will study the effects of rotation and tides on a self-gravitating sphere of fluid of uniform density ρ_0.

a) Consider the equilibrium of a roughly spherical body of fluid rotating homogeneously with angular velocity ω_0. Show that the effect of rotation can be accounted for by introducing an "effective gravitational potential"

$$\varphi_{\text{eff}} = \varphi_{\text{grav}} + \frac{1}{3}\omega_0^2 R^2 (P_2(\cos\theta) - 1),$$

where R, θ are spherical coordinates defined with their origin in the centre of the body and $\hat{z}$ along the axis of rotation.

b) A small planet is in a circular orbit about a distant massive star. It rotates about an axis perpendicular to the plane of the orbit so that it always keeps the same face directed towards the star. Show that the tidal forces can be taken into account by introducing an effective external potential

$$\varphi_{\text{tidal}} = -\Omega^2 R^2 P_2(\cos\theta),$$

together with a potential of the same sort as in part a) that accounts for the once-per-orbit rotation. Here Ω is the orbital angular velocity, and R, θ are spherical coordinates defined with their origin at the centre of the planet and $\hat{z}$ pointing at the star.

c) The external potentials slightly deform the initially spherical planet and the surface is given by

$$R(\theta, \phi) = R_0 + \eta P_2(\cos\theta).$$

Show that, to first order in η, this deformation does not alter the volume of the body. Observe that positive η corresponds to a prolate spheroid and negative η to an oblate one.

d) The gravitational field of the deformed spheroid can be found by approximating it as an undeformed homogeneous sphere of radius R_0, together with a thin spherical shell of radius R_0 and surface mass density $\sigma = \rho_0 \eta P_2(\cos\theta)$. Use the general axisymmetric solution

$$\varphi(R, \theta, \phi) = \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l(\cos\theta)$$

of Laplace's equation, together with Poisson's equation

$$\nabla^2 \varphi = 4\pi G \rho(\mathbf{r})$$

for the gravitational potential, to obtain expressions for φ_{shell} in the regions $R > R_0$ and $R \le R_0$.

e) The surface of the fluid will be an equipotential of the combined potentials of the homogeneous sphere, the thin shell, and the effective external potential of the tidal or centrifugal forces. Use this fact to find η (to lowest order in the angular velocities) for the two cases. Do not include the centrifugal potential from part b) when computing the purely tidal distortion. I only made the planet rotate synchronously in order to keep the the fluid stationary with respect to the tidal bulges.

(See: `https://tidesandcurrents.noaa.gov/restles3.html` for an explanation of why we ignore the spatial variation of the centrifugal force when computing tidal effects.)

2) An old Qual problem: A sphere of radius a is made by joining two conducting hemispheres along their equators. The hemispheres are electrically insulated from one another and maintained at two different potentials V_1 and V_2.

a) Starting from the general expression

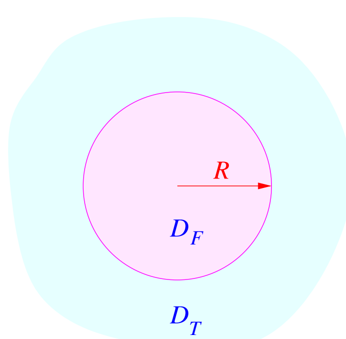
$$V(r,\theta) = \sum_{l=0}^{\infty} \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) P_l(\cos\theta)$$

find an integral expression for the coefficients a_l, b_l that are relevent to the electric field *outside* the sphere. Evaluate the integrals giving b_1, b_2 and b_3.

b) Use your results from part a) to compute the electric dipole moment of the sphere as function of the potential difference $V_1 - V_2$.

c) Now the two hemispheres are electrically connected and the entire surface is at one potential. The sphere is immersed in a uniform electric field $\mathbf{E}$. What is its dipole moment now?

2) Peierls Problem: Computing the Critical Mass. The core of a fast breeder reactor consists of a sphere of fissile ^{235}U of radius R. It is surrounded by a thick shell of non-fissile material which acts as a neutron reflector, or *tamper*.



Fast breeder reactor.

In the core, the fast neutron density $n(\mathbf{r}, t)$ obeys

$$\frac{\partial n}{\partial t} = \nu\, n + D_F \nabla^2 n.$$

Here the term with ν accounts for the production of additional neutrons due to induced fission. The term with D_F describes the diffusion of the fast neutrons. In the tamper the neutron flux obeys

$$\frac{\partial n}{\partial t} = D_T \nabla^2 n.$$

Both the neutron density, n, and the neutron flux, $\mathbf{j} = D_{F,T} \nabla n$, are continuous across the interface between the two materials. Find an equation determining the critical radius, R_c, above which the neutron density grows without bound. Show that the critical radius for an assembly with a tamper consisting of ^{238}U ($D_T = D_F$) is one-half of that for a core surrounded only by air ($D_T = \infty$), and so the use of a thick ^{238}U tamper reduces the critical mass by a factor of eight.

3) Bessel functions and impact parameters: In two dimensions we can expand a plane wave as

$$e^{iky} = \sum_{n=-\infty}^{\infty} J_n(kr) e^{in\theta}.$$

a) What do you think the resultant wave will look like if we take only a finite segment of this sum? For example

$$\phi(x) = \sum_{l=10}^{17} J_n(kr) e^{in\theta}.$$

Think about:

i) The quantum interpretation of $\hbar l$ as angular momentum $= \hbar k d$, where d is the *impact parameter*, the amount by which the incoming particle misses the origin.

ii) Diffraction: one cannot have a plane wave of finite width.

b) After writing down your best guess for the previous part, confirm your understanding by using Mathematica or other package to plot the real part of ϕ as defined above.