Physics 487 Midterm Exam #2 Spring 2021
Thursday April 29, 11:00 am – 12:20 pm

This is a closed book exam. No use of calculators or any other electronic devices is allowed except to view this file.

You have **75 minutes** to work the problems and
5 minutes to photograph your answers and upload using the my.physics course upload tool.

At the beginning of the exam:
1) Write your **name** and **netid** on your answer booklet(s).
2) Turn your **cell phone off**.
3) Put away all calculators, phones, computers, notes, and books.

During the exam:
1) **Show your work** and/or reasoning. **<u>Answers with no work or explanation get no points.</u>** But ...
2) **Don't write long essays** explaining your reasoning. We only need to see enough work to confirm that you **understand** what you're doing and are **not just guessing**. (If you *are* guessing, explain that, then *verify* your guess explicitly.) A good **annotated sketch** is often the best explanation of all!
3) **<u>All question parts on this exam are independent</u>**: you can get full points on any part even if your answers to all the other parts are incorrect. You should **attempt all the question parts!** If you get stuck, move on to the next one and come back later. The worst thing you can do is stall on one question and not get to others whose solution may be very simple.
4) Partial credit will be given for incorrect answers if the work is understandable and some of it is correct. IMPORTANT: If you think you've made a mistake but can't find it, **<u>explain what you think is wrong</u>** ➔ you may well get partial credit for noticing your error!
5) It is fine to leave answers as **radicals or irreducible fractions** (e.g. $10\sqrt{3}$ or 5/7), but you will lose points for not simplifying answers to an **irreducible form** (e.g. $24(x^2 - y^2)/(\sqrt{9}x - \sqrt{9}y)$ is unacceptable.)

Academic Integrity:

The giving of assistance to or receiving of assistance from another person, or the use of unauthorized materials during University Examinations can be grounds for disciplinary action, up to and including expulsion from the University.

Please remember: **"no work, no points".** Everything that isn't on the formula sheets 486-Final or 487-Midterm2 needs to be derived and/or explained.

You have **75 minutes** to work + 5 to upload
→ upload by 12:20 pm

Problem 1

A particle is located in this potential:

$$V(x) = A\left(\frac{\sin(x)}{x}\right)^2$$

(a) Write down a trial wavefunction that could be used with the variational principle (v.p.) to estimate the ground state of this system.

(b) Write down a trial wavefunction that could be used with the v.p. to estimate the first excited state.

(c) Write down a potential $V(x)$ that would make it impossible to estimate the first excited state using the v.p. and explain why you chose what you did.

Problem 2

Consider a quantum system with only three linearly-independent states. The Hamiltonian of the system is $H = H_0 + H'$ where H' is a perturbation that is very small compared to the unperturbed Hamiltonian H_0. Using as our basis the energy eigenstates $\{ |n^{(0)}\rangle \} = \{ |1^{(0)}\rangle, |2^{(0)}\rangle, |3^{(0)}\rangle \}$ of the unperturbed Hamiltonian, the matrix representations of H_0 and H' are :

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}' \quad \text{where} \quad \mathbf{H}_0 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{H}' = \varepsilon \begin{pmatrix} 1/2 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

"Appropriate units" are in use (i.e. ignore units), and the parameter ε multiplying the perturbation $\mathbf{H}'$ is a small dimensionless number << 1.

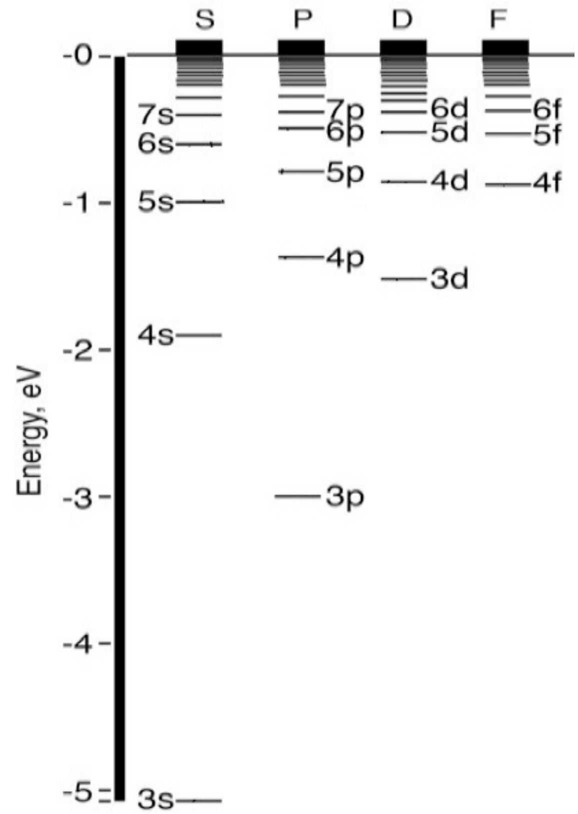
(a) Calculate the first eigenvalue E_1 of H to $0^{\text{th}} + 1^{\text{st}} + 2^{\text{nd}}$ order in ε.

(b) Calculate the first eigenstate $|1\rangle$ of H to $0^{\text{th}} + 1^{\text{st}}$ order in ε.

There are more questions on the next page.

Problem 3

An atom has one valence electron, whose ground state is the 3s orbital (because the orbitals below that are filled). The energy levels available to the electron are shown in the diagram at right.



(a) A researcher excites the electron to the 5s orbital. Identify all the routes that the electron can take to return to its 3s ground state via spontaneous E1 radiation.

(b) Is it a reasonable approximation to ignore E2, E3, … radiation when answering the above question? Throw some numbers together to estimate whether it is or isn't reasonable.

Problem 4

A particle of mass m and charge q sits in a 1D infinite well that runs from $x = 0$ to $x = L$ and has $V = 0$ within the well, as usual. The energy eigenstates of the particle are our familiar friends:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right).$$

The impenetrable walls of this ∞ well are capacitor plates. At time $t = 0$, they begin to charge up, which subjects the particle to the time-dependent perturbation

$$V(x,t) = qE_0 x\left(1 - e^{-t/\tau}\right), \quad \text{where } E_0 \text{ and } \tau \text{ are known constants.}$$

The particle was in its ground state at times $t \leq 0$. What is the probability that the electric force from the capacitor plates causes it to transition to the first excited state of the uncharged capacitor-well at times $t > 0$? You may assume that $V(x,t)$ is very small compared to the particle's kinetic energy. <u>If your answer involves any integral(s), set them up clearly and completely, but DO NOT EVALUATE THEM</u>, just assign them a symbol.

Problem 5

At times $t \leq 0$, an electron is sitting in the ground state of a hydrogen (Z=1) atom. At time $t = 0$, an Elven Mage casts a magic spell and instantly transforms the hydrogen nucleus to a helium nucleus (Z=2).

(a) An instant later, at time $t = 0+$, a Dwarven Cleric measures the principal quantum number n of the electron (which is now in a helium atom). What is the probability that the Dwarven Cleric obtains $n = 1$? <u>If your answer involves any integral(s), set them up clearly and completely, but DO NOT EVALUATE THEM</u>, just assign them a symbol.

(b) The Dwarven Cleric took an "instant" to make their measurement, by which we mean an arbitrarily small amount of time. Estimate how fast this "instant" would have to be for your answer to part (a) to be approximately accurate.

1D SHO $\hat{H}(x)=\frac{1}{2m}\left(\hat{p}^2+m^2\omega^2x^2\right)$ Define $x_0\equiv\sqrt{\frac{\hbar}{m\omega}}$, $\xi\equiv\frac{x}{x_0}$ $\rightarrow$ $\hat{H}(\xi)=\frac{\hbar\omega}{2}\left(\xi^2-\frac{d^2}{d\xi^2}\right)$

$E_n=\left(n+\tfrac{1}{2}\right)\hbar\omega$, $\psi_n(x)=\left(\frac{1}{\pi x_0^2}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{x}{x_0}\right)e^{-\frac{x^2}{2x_0^2}}$ with Hermite poly.: $H_0(\xi)=1$, $H_2(\xi)=4\xi^2-2$, $H_1(\xi)=2\xi$, $H_3(\xi)=8\xi^3-12\xi$,

$\hat{a}_\pm=\frac{1}{\sqrt{2}}\left(\xi\mp\frac{d}{d\xi}\right)$: $\hat{a}_+\psi_n=\sqrt{n+1}\,\psi_{n+1}$, $\hat{a}_-\psi_n=\sqrt{n}\,\psi_{n-1}$, $\hat{x}=x_0(\hat{a}_++\hat{a}_-)/\sqrt{2}$, $\hat{p}=i\hbar(\hat{a}_+-\hat{a}_-)/(\sqrt{2}x_0)$, $[\hat{a}_-,\hat{a}_+]=1$, $\hat{H}=\hbar\omega\left(\hat{a}_+\hat{a}_-+\tfrac{1}{2}\right)$, $H_n(\xi)=(-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J / M	J / M	...
m_1	m_2	Coefficients		
.	.			

$Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$

$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$Y_2^0=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$

$Y_2^1=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$

$Y_2^2=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$

$Y_\ell^{-m}=(-1)^m Y_\ell^{m*}$

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$

1/2×1/2

m_1 m_2	J=1, M=+1
+1/2 +1/2	1

m_1 m_2	J=1, M=0	J=0, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=1, M=−1
−1/2 −1/2	1

1×1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

2×1/2

m_1 m_2	5/2, +5/2
+2 +1/2	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

2×1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

1×1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

3/2×1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1

3/2×3/2

m_1 m_2	3, +3
+3/2 +3/2	1

m_1 m_2	3, +2	2, +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3, +1	2, +1	1, +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3, 0	2, 0	1, 0	0, 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3, −1	2, −1	1, −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3, −2	2, −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3, −3
−3/2 −3/2	1

2×3/2

m_1 m_2	7/2, +7/2
+2 +3/2	1

m_1 m_2	7/2, +5/2	5/2, +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	7/2, −5/2	5/2, −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2, −7/2
−2 −3/2	1

2×2

m_1 m_2	4, +4
+2 +2	1

m_1 m_2	4, +3	3, +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4, +2	3, +2	2, +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	4, +1	3, +1	2, +1	1, +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4, −1	3, −1	2, −1	1, −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	4, −2	3, −2	2, −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4, −3	3, −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4, −4
−2 −2	1

Atomic Structure

Bohr magneton: $\mu_B=\frac{e\hbar}{2m_e}$

gyromag. ratio γ: $\vec{\mu}_J=\gamma\vec{J}$, $\gamma_{\text{classical}}=\frac{e}{2m}$

g factor: $\vec{\mu}_L=\frac{e}{2m}\vec{L}$, $\vec{\mu}_S=g\frac{e}{2m}\vec{S}$, $g_{\text{spin-1/2 point particle}}=2$

Hund rules: 1. Max S 2. Max L 3. Min J for $\leq$ 1/2-filled shells

$l=$ 0 1 2 3 4... ; s p d f g...   term symbol: ${}^{2S+1}L_J$

Perturbation Theory – Time-Independent

$H = H_0 + H'$

- H_0 solvable w eigen-* $\{E_n^{(0)}\},\{|n^{(0)}\rangle\}$
- $H' \ll H_0$

Expansions for eigen-* of H: $E_n = E_n^{(0)} + E_n^{(1)} + \cdots$ & $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \cdots$

For a **non-degenerate** eigenvalue $E_n^{(0)}$ of H_0: $|n^{(1)}\rangle = \sum_{m\neq n} \frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$ with $H'_{mn} \equiv \langle m^{(0)}|H'|n^{(0)}\rangle$

$$E_n^{(j)} = \langle n^{(0)}|H'|n^{(j-1)}\rangle \quad \rightarrow \quad E_n^{(1)} = H'_{nn}, \quad E_n^{(2)} = \sum_{m\neq n} \frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

For a **degenerate** eigenvalue $E_D^{(0)}$ of H_0:

- Let $\{ |\alpha_1^{(0)}\rangle, ..., |\alpha_n^{(0)}\rangle \}$ = degen. subspace D sharing e-value $E_D^{(0)}$
- Find $\{ |\beta_1^{(0)}\rangle, ..., |\beta_n^{(0)}\rangle \}$ = e-vectors of H' within subspace D
  = linear combinations of $|\alpha_i^{(0)}\rangle$ states that diagonalize $\mathbf{H'}$

$\Rightarrow$ 1st order energy correction is $E_{\beta_i}^{(1)} = \langle \beta_i^{(0)}|H'|\beta_i^{(0)}\rangle$

Variational Principle

$E_{gs} \leq \langle\psi|H|\psi\rangle \;\forall\; \psi$

Sudden / Adiabatic Approx

ψ / n unchanged by ΔH

Perturbation Theory – Time Dependent

- $H(t) = H^{(0)} + H'(t)$
- $\{ E_n^{(0)}, |n^{(0)}\rangle \}$ = the eigen-* of $H^{(0)}$
- $\omega_{fn} \equiv \left(E_f^{(0)} - E_n^{(0)}\right)/\hbar$
- $H'_{fn} \equiv \langle f^{(0)}|H'|n^{(0)}\rangle$

$$|\psi(t)\rangle = \sum_n c_n(t)\, e^{-i\omega_n t} |n^{(0)}\rangle \quad \text{where} \quad i\hbar \dot{c}_f(t) = \sum_n H'_{fn}\, e^{i\omega_{fn} t} c_n(t)$$

To 1st order in $H' \ll H^{(0)}$, with $|\psi(t_0)\rangle = |i^{(0)}\rangle$: $c_f(t) \approx \delta_{fi} + \frac{1}{i\hbar}\int_{t_0}^{t} H'_{fi}(t')\, e^{i\omega_{fi} t'}\, dt' \;\rightarrow\; P_{i\to f} = |c_f(t)|^2$

relevant math for analyzing time- & frequency-dependence: $\frac{\sin(x)}{x} \xrightarrow[x\to 0]{} 1$, $\frac{\sin^2(ax)}{a x^2} \xrightarrow[a\to\infty]{} \pi\,\delta(x)$

Fermi's Golden Rule: $W_{i\to f} \equiv \frac{P_{i\to f}}{t} = \frac{2\pi}{\hbar}|V_{fi}|^2 n(E_f)$ at resonance

$E_f = E_i \pm \hbar\omega$ for $H' = V(r)\left(e^{i\omega t} + e^{-i\omega t}\right)$

$E_f = E_i$ for $H' = V(r)\,\Theta(t)$

E1 radiation: when $\lambda \gg r$ and F_B negligible, $H' = V(\vec{r})\cos(\omega t)$, $V(\vec{r}) \approx -q\vec{E}_0 \cdot \vec{r}$ $\rightarrow$ E1 selection rules:

For the electron making the E1 transition

(a) $\Delta l = \pm 1$

(b) $\Delta m_l = 0, \pm 1$

(c) spin unchanged: $\Delta m_s = 0$

For the atom as a whole

(a) $\Delta S = 0$

(b) $\Delta L = 0, \pm 1$ ($L = 0 \leftrightarrow L' = 0$ forbidden)

(c) $\Delta M_L = 0, \pm 1$

(d) $\Delta J = 0, \pm 1$ ($J = 0 \leftrightarrow J' = 0$ forbidden)

(e) $\Delta M_J = 0, \pm 1$

spontaneous emission rate = Einstein's $A_{i\to f} = \frac{\omega_{if}^3\, q^2 |\vec{r}_{fi}|^2}{3\pi\varepsilon_0 \hbar c^3}$ with $\vec{r}_{fi} \equiv \langle f^{(0)}|\vec{r}|i^{(0)}\rangle$

lifetime $\tau_i = \frac{1}{\sum_f A_{i\to f}}$

Old Quantum Theory (1900–1925)

$E = hf = \hbar\omega$, $p = h/\lambda = \hbar k$

Quantization Rules : $E = nh$, $\oint_{\text{one period}} p_q \cdot dq = n_q h$

Correspondence Principle : CM is recovered in the limit of large quantum #s ($n \to \infty$)

Probability and some 3D Calculus

for a probability distribution $P(x)$: mean $\langle x \rangle = \int_{x_{\min}}^{x_{\max}} P(x)\, x\, dx$, variance $\sigma_x^2 \equiv \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle = \langle x^2 \rangle - \langle x \rangle^2$, $\sigma_x \equiv$ standard deviation

3D operators in Cartesian coord's : $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ $\quad \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ $\quad \nabla^2 \equiv \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

GGS Theorems : $\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$ $\quad \int_{\text{Surf}} \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{A} = \oint_{\partial\text{Surf}} \vec{E}\cdot d\vec{l}$ $\quad \int_{\text{Vol}} \left(\vec{\nabla}\cdot\vec{E}\right) dV = \oint_{\partial\text{Vol}} \vec{E}\cdot d\vec{A}$

Wave Mechanics

The **inner product** of two wavefunctions f & g : $\langle f | g \rangle \equiv \int_{-\infty}^{+\infty} f(\vec{r})^*\, g(\vec{r})\, d^3\vec{r}$

Physical observables Q correspond to **Hermitian operators** $\hat{Q}$ ≡ linear operators with this defining property

(presented in three equivalent forms) : 1. $\langle Q \rangle^* = \langle Q \rangle$ 2. $\langle \Psi | \hat{Q}\Psi \rangle = \langle \hat{Q}\Psi | \Psi \rangle$ i.e. $\hat{Q}$ is **self-adjoint**

3. eigenstates of $\hat{Q}$ are complete over their Hilbert space

Schrödinger Equation : $\hat{H}\Psi = i\hbar\frac{\partial \Psi}{\partial t}$

Operators in 3D $\vec{r}$-space : $\hat{\vec{p}} = \frac{\hbar}{i}\vec{\nabla}$, $\hat{\vec{r}} = \vec{r}$, $\hat{H} = \frac{\hat{p}^2}{2m} + V = -\frac{\hbar^2\nabla^2}{2m} + V(\vec{r})$

Eigenfunctions of $\hat{p}, \hat{x}$ with Dirac normalization: $\psi_p(x) = e^{ipx/\hbar}/\sqrt{2\pi\hbar}$, $\psi_{x'}(x) = \delta(x - x')$

Boundary Conditions on wavefunctions:

a. Wavefunctions are always **continuous.**
b. Wavefunctions have **continuous derivatives**, **except** at points where $V = \pm\infty$
   where $\lim_{\varepsilon\to 0} \psi'(x+\varepsilon) - \psi'(x-\varepsilon) = \left(2m/\hbar^2\right) \lim_{\varepsilon\to 0} \int_{x-\varepsilon}^{x+\varepsilon} V(x)\psi(x)\, dx$
c. Wavefunctions are **zero** in any region where $V = \infty$.

Probability density $\rho(\vec{r},t)$ $= |\Psi(\vec{r},t)|^2 = \Psi^*\Psi$

Prob. current density $\vec{j}(\vec{r},t) = \mathrm{Re}\left[\Psi^* \frac{\hat{\vec{p}}}{m}\Psi\right]$

Continuity Equation : $-\frac{\partial \rho}{\partial t} = \vec{\nabla}\cdot\vec{j}$ $\quad R,T = \frac{\vec{j}_{\text{re,tr}}\cdot\vec{A}}{\vec{j}_{\text{in}}\cdot\vec{A}}$

Expectation Value $\langle Q \rangle$ of observable $Q(\vec{r},\vec{p})$: $\langle Q \rangle \equiv \langle \Psi | \hat{Q}\Psi \rangle \equiv \int_{-\infty}^{+\infty} \Psi^*\, \hat{Q}\left(\vec{r}, -i\hbar\vec{\nabla}\right)\Psi\, d^3\vec{r}$

Ehrenfest's Theorem : Expectation values follow classical laws. $\frac{\langle p \rangle}{m} = \frac{d\langle x \rangle}{dt}$, $\frac{d\langle p \rangle}{dt} = \left\langle -\frac{dV}{dx} \right\rangle$

Virial Theorem : $2\langle T \rangle = \left\langle x\frac{dV}{dx} \right\rangle$

Representations of a state $|\psi\rangle$ & operator $\hat{A}$: In the eigenbasis $\{|e_q\rangle\}$ of any Hermitian operator $\hat{Q}$,

- **Wavefuncⁿ** repres : $\langle f | g \rangle = \int \vec{f}^*(q)\vec{g}(q)\, dq$
  wavefunction $\psi(q) = \langle e_q | \psi \rangle$
  & differential operator $\hat{A}(q, \frac{\partial}{\partial q}, \frac{\partial^2}{\partial q^2}, \ldots)$
- **Matrix** repres : inner product $\langle f | g \rangle = \vec{f}^{*T}\vec{g}$
  column vector $\vec{\psi} = \begin{pmatrix} \langle e_1 | \psi \rangle \\ \langle e_2 | \psi \rangle \\ \ldots \end{pmatrix}$ & matrix with elements $\mathbf{A}_{ij} = \langle e_i | \hat{A} | e_j \rangle$

e.g. wavefunction conversion between x- and p-space : $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{ipx/\hbar}\phi(p)\, dp \iff \phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\psi(x)\, dx$

e.g. Operators in 1D p-space : $\hat{p} = p$, $\hat{x} = i\hbar\frac{\partial}{\partial p}$, $\hat{H} = \frac{p^2}{2m} + V\left(i\hbar\frac{\partial}{\partial p}\right)$

Commutator : $\left[\hat{A},\hat{B}\right]\equiv\hat{A}\hat{B}-\hat{B}\hat{A}$ Theorem : Operators that commute share a common set of eigenstates.

Uncertainty Principle : $\sigma_A\sigma_B \geq \left|\frac{1}{2i}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|$ e.g. $\sigma_x\sigma_p \geq \frac{\hbar}{2}$

Time-dep. of Expec. Value : $\frac{d\langle\hat{Q}\rangle}{dt}=\frac{i}{\hbar}\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle+\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$

Axioms of QM

1. The **STATE** of a QM system is represented by a vector $|\Psi(t)\rangle$ in a Hilbert space (≈ Inner Product Space).

2. **OBSERVABLES** Q are represented by Hermitian operators $\hat{Q}$. In x-space ≡ the eigenbasis of the position operator $\hat{x}$, the phase space operators are $\hat{x}=x$ & $\hat{p}_x=\frac{\hbar}{i}\frac{\partial}{\partial x}$, and those of dependent observables are $\hat{Q}(\hat{x},\hat{p})$.

3. **MEASUREMENT** of an observable Q will yield one of its eigenvalues q, and the state of the system will change from $|\psi\rangle$ to the corresponding eigenstate $|e_q\rangle$. Allowed eigenstates are constrained by physical requirements such as boundary conditions and normalizability.

4. The **PROBABILITY** of measuring a particular eigenvalue q from a state $|\psi\rangle$ is $P(q)=\left|\langle e_q|\psi\rangle\right|^2$.

5. The **TIME-EVOLUTION** of a quantum state is given by the Schrödinger Equation, $i\hbar\frac{d}{dt}|\Psi(t)\rangle=\hat{H}|\Psi(t)\rangle$.

6. A multiparticle state containing two **IDENTICAL PARTICLES** is symmetric/anti-symmetric under their exchange if the particles are bosons (integer spin) / fermions (half-integer spin).

Miscellaneous Math

Gaussian prob dist^n : $P(x;x_0,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-x_0)^2/2\sigma^2}$ Sums : $\sum_{j=0}^{\mu}1=\mu+1$, $\sum_{j=0}^{\mu}j=(\mu+1)\frac{\mu}{2}$

Gaussian Integrals: $\int_{-\infty}^{+\infty}e^{-ax^2-bx}\,dx=\sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}}$ $\quad\int_{-\infty}^{+\infty}x\,e^{-ax^2-bx}\,dx=-\frac{\sqrt{\pi}\,b}{2a^{3/2}}\,e^{\frac{b^2}{4a}}$ $\quad\int_{-\infty}^{+\infty}x^2\,e^{-ax^2-bx}\,dx=\frac{\sqrt{\pi}}{4a^{5/2}}\left(2a+b^2\right)e^{\frac{b^2}{4a}}$

Exponential Integrals: $\int_0^{\infty}x^n\,e^{-x}\,dx=\Gamma(n+1)=n!$ $\quad\int x\,e^{-ax}\,dx=-\frac{e^{-ax}}{a^2}(ax+1)$ $\quad\int x^2\,e^{-ax}\,dx=-\frac{e^{-ax}}{a^3}\left(a^2x^2+2ax+2\right)$

Sinusoidal Integrals: $\int_0^{\pi}\begin{matrix}\sin^2(a\phi)\\ \cos^2(a\phi)\end{matrix}\,d\phi=\frac{\pi}{2}\mp\frac{\sin(2\pi a)}{4a}$ $\quad\int_0^{\pi}\begin{matrix}\sin(n\phi)\sin(m\phi)\\ \cos(n\phi)\cos(m\phi)\end{matrix}\,d\phi=\delta_{nm}$ $\quad\int_0^{\pi}\sin(n\phi)\cos(m\phi)\,d\phi=0$

Fourier Integrals: $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A(k)e^{ikx}\,dk$ where $A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{-ikx}\,dx$

Dirac δ function : $\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{iqx}\,dq$

Defining Properties :
1. $\delta(x)=0$ when $x\neq 0$
2. $\delta(x)=\infty$ when $x=0$ OR $\int_{-\infty}^{+\infty}\delta(x)f(x)dx=f(0)$
3. $\int_{-\infty}^{+\infty}\delta(x)dx=1$

Classical Mechanics security blanket ☺

$L(q_i,\dot{q}_i,t)=T-U$ Lagrange EOM: $\frac{\partial L}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)$

$H\equiv\dot{q}_i(\partial L/\partial\dot{q}_i)-L$ equals $T+U$ when $\vec{r}_a=\vec{r}_a(q_i)$

$dH/dt=-\partial L/\partial t$

Common Forces : $F_{\text{grav}}=\frac{Gm_1m_2}{r^2}$, $F_{\text{elec}}=\frac{q_1q_2}{4\pi\varepsilon_0 r^2}$, $F_{\text{cf}}=\frac{mv^2}{r}$

Generalized momentum $p_i\equiv\frac{\partial L}{\partial\dot{q}_i}$, force $Q_i\equiv\frac{\partial L}{\partial q_i}$

Hamilton's EOM: $-\frac{\partial H}{\partial q_i}=\frac{dp_i}{dt}$, $\frac{\partial H}{\partial p_i}=\frac{dq_i}{dt}$

Special Relativity: $E^2=(pc)^2+(mc^2)^2$

$\gamma=\frac{1}{\sqrt{1-(v/c)^2}}$, $E=\gamma mc^2$, $p=\gamma mv$, $v=\frac{pc^2}{E}$

Constants : $m_e c^2 = 0.511$ MeV $\quad \hbar c \begin{array}{l} = 197 \text{ MeV}\cdot\text{fm} \\ \approx 200 \text{ MeV}\cdot\text{fm}\end{array} \quad \alpha \equiv \dfrac{e^2}{4\pi\varepsilon_0 \hbar c} = \dfrac{1}{137} \quad a_0 \equiv \dfrac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = \dfrac{(\hbar c)}{\alpha\,(m_e c^2)}$

Angular Momentum $\quad \hat{L}^2 = \left|\vec{r}\times\dfrac{\hbar}{i}\vec{\nabla}\right|^2 = -\hbar^2\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}\right]$

$$\hat{L}_x = +i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right), \quad \hat{L}_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right), \quad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

Spin & Angular Momentum : L, l can be replaced by S, s $\quad [L^2, L_{x,y,z}] = 0$, $\quad \left[L_x, L_y\right] = i\hbar L_z$, etc

$$L^2|l\,m\rangle = \hbar^2 l(l+1)|l\,m\rangle, \quad L_z|l\,m\rangle = \hbar m|l\,m\rangle, \quad L_\pm = L_x \pm iL_y, \quad L_\pm|l\,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,|l(m\pm1)\rangle$$

Pauli Spin Matrices $\left\{S_x, S_y, S_z\right\} = \dfrac{\hbar}{2}\left\{\sigma_x, \sigma_y, \sigma_z\right\}$ where $\sigma_x, \sigma_y, \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

Spherical Harmonics $Y_l^m(\theta,\phi)$, $m = -l,\ldots,l$ in steps of 1

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \qquad Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm2i\phi}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$$

$$Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi} \qquad Y_3^{\pm1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \qquad Y_3^{\pm2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm2i\phi}$$

$$Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \qquad Y_3^{\pm3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm3i\phi}$$

H-like atom : radial e-functions $R_{nl}(r)$

$Ze \equiv$ nuclear charge (Z=1 is hydrogen)

$$R_{10} = 2\left(\frac{Z}{a_0}\right)^{3/2}\exp\left(-\frac{Zr}{a_0}\right)$$

$$R_{20} = \left(\frac{Z}{2a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right)\exp\left(-\frac{Zr}{2a_0}\right)$$

$$R_{21} = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)\exp\left(-\frac{Zr}{2a_0}\right)$$

$$E_n = -\frac{(Z\alpha)^2}{2n^2}\left(m_e c^2\right) \text{ for } n = 1,2,3,\ldots$$

Spherical Coordinates $\qquad$ *Line Element:* $d\vec{l} = dr\ \hat{r} + r\,d\theta\ \hat{\theta} + r\sin\theta\,d\phi\ \hat{\phi}$

$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi\ \hat{r} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}$$

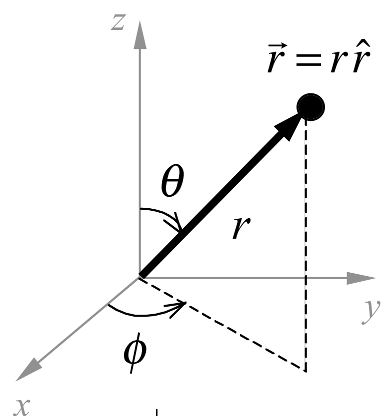
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi\ \hat{r} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}$$

$$z = r\cos\theta \qquad \hat{z} = \cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$

$$\theta = \tan^{-1}(\sqrt{x^2+y^2}\,/\,z) \qquad \hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$$

$$\phi = \tan^{-1}(y\,/\,x) \qquad \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$



Gradient: $\quad \vec{\nabla}V = \dfrac{\partial V}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial V}{\partial\theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial V}{\partial\phi}\hat{\phi}$

	∂_r	∂_θ	∂_ϕ
$\hat{r}$	0	$\hat{\theta}$	$\sin\theta\hat{\phi}$
$\hat{\theta}$	0	$-\hat{r}$	$\cos\theta\hat{\phi}$
$\hat{\phi}$	0	0	$-\sin\theta\hat{r}$ $-\cos\theta\hat{\theta}$

Laplacian: $\quad \nabla^2 V = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 V}{\partial\phi^2}$

Divergence: $\quad \vec{\nabla}\cdot\vec{E} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 E_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\ E_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial E_\phi}{\partial\phi}$

Curl: $\quad \vec{\nabla}\times\vec{E} = \dfrac{\hat{r}}{r\sin\theta}\left[\dfrac{\partial}{\partial\theta}(\sin\theta\ E_\phi) - \dfrac{\partial E_\theta}{\partial\phi}\right] + \dfrac{\hat{\theta}}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial E_r}{\partial\phi} - \dfrac{\partial}{\partial r}(rE_\phi)\right] + \dfrac{\hat{\phi}}{r}\left[\dfrac{\partial}{\partial r}(rE_\theta) - \dfrac{\partial E_r}{\partial\theta}\right]$

Acceleration: $\quad \vec{a} = \hat{r}\left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right] + \hat{\theta}\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right] + \hat{\phi}\left[\sin\theta(r\ddot{\phi} + 2\dot{r}\dot{\phi}) + \cos\theta(2r\dot{\theta}\dot{\phi})\right]$

$$|\vec{v}| \equiv \sqrt{\vec{v}\cdot\vec{v}} \qquad \vec{v} = \sum_{i=1}^{3}(\vec{v}\cdot\hat{r}_i)\,\hat{r}_i \qquad d\vec{l}_{path} = \frac{d\vec{l}}{du}\,du \qquad d\vec{A} = \left(\frac{\partial\vec{l}}{\partial u}\times\frac{\partial\vec{l}}{\partial v}\right)du\,dv$$

$$df(x_1,\ldots,x_n) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\,dx_i \qquad dV = \left(\frac{\partial\vec{l}}{\partial u}\times\frac{\partial\vec{l}}{\partial v}\right)\cdot\frac{\partial\vec{l}}{\partial w}\,du\,dv\,dw$$

Conceptual version:

$$d\vec{l}_u \equiv \frac{\partial\vec{l}}{\partial u}du$$

$$d\vec{l}_{path} = d\vec{l}_u$$

$$d\vec{A} = d\vec{l}_u \times d\vec{l}_v$$

$$dV = (d\vec{l}_u \times d\vec{l}_v)\cdot d\vec{l}_w$$

Taylor

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1st order approx for $x \ll 1$:

- $(1+x)^n \approx 1+nx$
- $\sin x \approx x$
- $\sin^{-1} x \approx x$
- $\cos x \approx 1 - \frac{x^2}{2}$
- $\cos^{-1} x \approx \frac{\pi}{2} - x$
- $\tan x \approx x$
- $\tan^{-1} x \approx x$
- $e^x \approx 1+x$
- $\ln(1+x) \approx x$

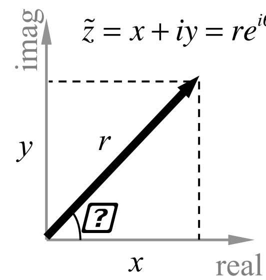
	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

$$\sin a\,\sin b = \tfrac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$
$$\cos a\,\cos b = \tfrac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$$
$$\sin a\,\cos b = \tfrac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$

Complex Numbers

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\tilde{z} = x + iy = re^{i\theta}$$



$$\tilde{z}^* \equiv x - iy = re^{-i\theta}$$

$$|\,\tilde{z}\,| \equiv \sqrt{\tilde{z}^*\tilde{z}} = r$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin(a+b) = \sin a\cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a \sin b$$

Integral Table

$$\int\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right)$$

$$\int\frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left(x+\sqrt{x^2\pm a^2}\right)$$

$$\int\frac{x\;dx}{\sqrt{a^2\pm x^2}} = \pm\sqrt{a^2\pm x^2}$$

$$\int_0^{2\pi}\sin^2\phi\;d\phi = \int_0^{2\pi}\cos^2\phi\;d\phi = \pi$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int\frac{x\;dx}{(a\pm x)^2} = \frac{a}{a\pm x} + \ln(a\pm x)$$

$$\int\sin^2\phi\;d\phi = \frac{\phi}{2} - \frac{\sin(2\phi)}{4}$$

$$\int\frac{dx}{(a\pm x)^2} = \mp\frac{1}{a\pm x}$$

$$\int\frac{x\;dx}{a^2\pm x^2} = \pm\frac{1}{2}\ln\left(a^2\pm x^2\right)$$

$$\int\cos^2\phi\;d\phi = \frac{\phi}{2} + \frac{\sin(2\phi)}{4}$$

$$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int\frac{x\;dx}{\left(a^2\pm x^2\right)^{3/2}} = \mp\frac{1}{\sqrt{a^2\pm x^2}}$$

$$\int\sin^3\theta\;d\theta = \frac{\cos^3\theta}{3} - \cos\theta$$

$$\int\frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$$

$$\int\ln(ax)\;dx = x\ln(ax) - x$$

$$\int\cos^3\theta\;d\theta = \sin\theta - \frac{\sin^3\theta}{3}$$

$$\int\cos^n\theta\,\sin\theta\;d\theta = -\frac{\cos^{n+1}\theta}{n+1}$$

$$\int\frac{dx}{(a^2\pm x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2\pm x^2}}$$

$$\int\frac{\ln(ax)}{x}\;dx = \frac{1}{2}\left[\ln(ax)\right]^2$$

$$\int\sqrt{a^2-x^2}\;dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\int\frac{x^2}{\sqrt{a^2-x^2}}\;dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\int\sqrt{x^2\pm a^2}\;dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int\frac{(x-a\cos\theta)\,\sin\theta\;d\theta}{(x^2+a^2-2ax\cos\theta)^{3/2}} = \frac{1}{x^2}\,\frac{a-x\cos\theta}{\sqrt{x^2+a^2-2ax\cos\theta}}$$