Lecture 19
Addition of Angular Momentum

Addition of Angular Momentum: Spin-1/2

We now turn to the question of the addition of angular momenta. This will apply to both spin and orbital angular momenta, or a combination of the two.

Suppose we have two spin-½ particles whose spins are given by the operators S_1 and S_2. The relevant commutation relations are

$$\left[S_{1x},S_{1y}\right]=i\hbar S_{1z} \text{ etc.}$$
$$\left[S_{2x},S_{2y}\right]=i\hbar S_{2z} \text{ etc.}$$
$$\left[\hat{S}_1,\hat{S}_2\right]=0$$

where the last one refers to any components of S_1 and S_2 and is zero because the degrees of freedom of the two particles are completely independent (i.e. S_1 doesn't operate on particle 2, and vice versa). We define the total spin of the two-particle system by

$$\hat{S}=\hat{S}_1+\hat{S}_2$$

The commutation relations for $\hat{S}$ are

$$\begin{aligned}\left[S_x,S_y\right]&=\left[S_{1x}+S_{2x},S_{1y}+S_{2y}\right]\\&=\left[S_{1x},S_{1y}\right]+\left[S_{2x},S_{2y}\right]+0+0\\&=i\hbar\left(S_{1z}+S_{2z}\right)=i\hbar S_z \text{ etc.}\end{aligned}$$

Therefore $\hat{S}$ satisfies the canonical angular momentum commutation relation, so we are justified in our definition of $\hat{S}$ as the total angular momentum operator. For a pair of spin-½ particles, there are four possible states for the complete system, which we label

$$\chi_+^{(1)}\chi_+^{(2)} \quad \chi_+^{(1)}\chi_-^{(2)} \quad \chi_-^{(1)}\chi_+^{(2)} \quad \chi_-^{(1)}\chi_-^{(2)}$$

where the χ's denote the two-component spinors, and the upper label is the particle number and the lower label corresponds to the projection of the spin operator for that particle along some axis being either $\pm\hbar/2$.

The spinors $\chi^{(1,2)}$ satisfy

$$S_1^2\chi_\pm^{(1)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\chi_\pm^{(1)}$$
$$S_{1z}\chi_\pm^{(1)} = \pm\frac{\hbar}{2}\chi_\pm^{(1)}$$

and similarly for $\chi^{(2)}$ and S_2, but note that S_2 does not operate on $\chi^{(1)}$ and S_1 does not operate on $\chi^{(2)}$. Let's check the eigenvalues of S_z for the four states

$$S_z\chi_\pm^{(1)}\chi_\pm^{(2)} = \left(S_{1z}+S_{2z}\right)\chi_\pm^{(1)}\chi_\pm^{(2)}$$
$$= \left(S_{1z}\chi_\pm^{(1)}\right)\chi_\pm^{(2)} + \chi_\pm^{(1)}\left(S_{2z}\chi_\pm^{(2)}\right)$$

Each term in parentheses on the right hand side gives $\pm\hbar/2$. Therefore we have

$$S_z\chi_+^{(1)}\chi_+^{(2)} = \hbar\chi_+^{(1)}\chi_+^{(2)}$$
$$S_z\chi_+^{(1)}\chi_-^{(2)} = S_z\chi_-^{(1)}\chi_+^{(2)} = 0$$
$$S_z\chi_-^{(1)}\chi_-^{(2)} = -\hbar S_z\chi_-^{(1)}\chi_-^{(2)}$$

There is one state with $m_s = +1$, one with $m_s = -1$ and two with $m_s = 0$. These states can be grouped together into a *triplet* and a *singlet*. To help see how, let's define the raising and lowering operator for the total spin

$$S_\pm = S_{1\pm} + S_{2\pm}$$

and recall

$$S_\pm|sm\rangle = \text{h}\sqrt{s(s+1) - m(m\pm1)}\,|s\,m\pm1\rangle$$

so

$$S_{(1,2)-}\chi_+^{(1,2)} = \hbar\chi_-^{(1,2)}$$

Now we apply S_- to the $m_s = 1$ state

$$S_-\chi_+^{(1)}\chi_+^{(2)} = \left(S_{1-}\chi_+^{(1)}\right)\chi_+^{(2)} + \chi_+^{(1)}\left(S_{2-}\chi_+^{(2)}\right)$$
$$= \hbar\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right) \qquad m_s = 0$$

We can apply S_- again, remembering that

$$S_{(1,2)-}\chi_-^{(1,2)} = 0$$

which gives

$$\begin{aligned} S_-\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_-^{(1)}\chi_+^{(2)}\right) &= \left(S_{1-}\chi_-^{(1)}\right)\chi_+^{(2)} + \chi_-^{(1)}\left(S_{2-}\chi_+^{(2)}\right) + \left(S_{1-}\chi_+^{(1)}\right)\chi_-^{(2)} + \chi_+^{(1)}\left(S_{2-}\chi_-^{(2)}\right) \\ &= \left\{0 + \hbar\chi_-^{(1)}\chi_-^{(2)} + \hbar\chi_-^{(1)}\chi_-^{(2)} + 0\right\} \\ &= 2\hbar\chi_-^{(1)}\chi_-^{(2)} \qquad\qquad m_s = -1 \end{aligned}$$

We have stepped down two times from the $m_s = 1$ state. If we apply S_- a third time, we get zero, so this must be the lowest rung on the ladder. Thus we have three states which, when normalized properly are

$$\begin{aligned} & && |SM\rangle \\ &\chi_+^{(1)}\chi_+^{(2)} &\rightarrow\ & |11\rangle \\ &\frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}\right) &\rightarrow\ & |10\rangle \\ &\chi_-^{(1)}\chi_-^{(2)} &\rightarrow\ & |1-1\rangle \end{aligned}$$

Since $m_s = -1, 0, 1$ for these three, they must have $S = 1 = S_1 + S_2 \rightarrow$ they are the *triplet states.*

If you've been keeping track, you will have noticed that there is one leftover $m_s = 0$ state. This state has to go with a total spin of $S = 0 = |S_1 - S_2|$, the *singlet state*. This state is constructed to be orthogonal to the triplet $m_s = 0$ state and is

$$\frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}\right)$$

That's all fine, but how do we *know* that this state doesn't belong with the three triplet states above? Let's check the eigenvalue of the total spin squared operator S^2.

$$\begin{aligned} \hat{S}^2 &= \left(S_1 + S_2\right)^2 \\ &= S_1^2 + S_2^2 + 2S_1 \bullet S_2 \\ S_1 \bullet S_2 &= S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \end{aligned}$$

We are dealing with eigenstates of S_i^2 and S_{iz}, so we want to convert the dot product into operators whose action on these states is known. As usual, this means raising and lowering operators

$$S_{1+}S_{2-} = \left(S_{1x} + iS_y\right)\left(S_{2x} - iS_{2y}\right)$$
$$= S_{1x}S_{2x} + S_{1y}S_{2y} + iS_{1y}S_{2x} - iS_{1x}S_{2y}$$
$$S_{1-}S_{2+} = \left(S_{1x} - iS_y\right)\left(S_{2x} + iS_{2y}\right)$$
$$= S_{1x}S_{2x} + S_{1y}S_{2y} - iS_{1y}S_{2x} + iS_{1x}S_{2y}$$

therefore

$$S_{1+}S_{2-} + S_{1-}S_{2+} = 2\left(S_{1x}S_{2x} + S_{1y}S_{2y}\right)$$

and

$$2S_1 \bullet S_2 = 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

which gives

$$S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

Now we can check the two $m = 0$ states, let's call them $X_\pm$

$$X_\pm = \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} \pm \chi_-^{(1)}\chi_+^{(2)}\right)$$

We have

$$S_1^2 X_\pm = \frac{1}{\sqrt{2}}\left(\left(S_1^2\chi_+^{(1)}\right)\chi_-^{(2)} \pm \left(S_1^2\chi_-^{(1)}\right)\chi_+^{(2)}\right)$$
$$= S_1\left(S_1+1\right)\hbar^2\chi_+^{(1)}\chi_-^{(2)} \pm S_1\left(S_1+1\right)\hbar^2\chi_-^{(1)}\chi_+^{(2)}$$
$$= \frac{3}{4}\hbar^2\left(\frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} \pm \chi_-^{(1)}\chi_+^{(2)}\right)\right)$$
$$= \frac{3}{4}\hbar^2 X_\pm$$

similarly

$$S_2^2 X_\pm = \frac{3}{4}\hbar^2 X_\pm$$

The next term is

$$2S_{1z}S_{2z}X_\pm = 2\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)X_\pm$$
$$= -\frac{1}{2}\hbar^2 X_\pm$$

Finally, the last two terms are

$$\begin{aligned}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)X_{\pm} &= \frac{1}{\sqrt{2}}\left(S_{1+}\chi_{+}^{(1)}S_{2-}\chi_{-}^{(2)}+S_{1-}\chi_{+}^{(1)}S_{2+}\chi_{-}^{(2)}\pm S_{1+}\chi_{-}^{(1)}S_{2-}\chi_{+}^{(2)}+S_{1-}\chi_{-}^{(1)}S_{2+}\chi_{+}^{(2)}\right)\\ &= \frac{1}{\sqrt{2}}\left(0+\hbar^2\chi_{-}^{(1)}\chi_{+}^{(2)}\pm\hbar^2\chi_{+}^{(1)}\chi_{-}^{(2)}\pm 0\right)\\ &= \frac{\hbar^2}{\sqrt{2}}\left(\chi_{-}^{(1)}\chi_{+}^{(2)}\pm\chi_{+}^{(1)}\chi_{-}^{(2)}\right)\\ &= \pm\hbar^2 X_{\pm}\end{aligned}$$

Putting all these things together, we get

$$\begin{aligned}S^2X_{\pm} &= \hbar^2\left(\frac{3}{4}+\frac{3}{4}-\frac{1}{2}\pm 1\right)X_{\pm}\\ &\equiv S(S+1)\hbar^2X_{\pm}\end{aligned}$$

Therefore, the two states, corresponding to the ± sign above, indeed have $S=1,0$. The X_{+} state corresponds to $S=1$, and the X_{-} state to $S=0$, as claimed.

Addition of Angular Momentum: General Spin

What we have done in adding two spin ½'s together

$$\frac{1}{2}+\frac{1}{2} \quad\rightarrow\quad 1,0$$

is a special case of the more general problem of addition of angular momentum. In general,

$$\vec{J}=\vec{J}_1+\vec{J}_2$$

where $\vec{J}_1$ and $\vec{J}_2$ can describe the orbital angular momentum, the spin, or the total angular momentum of a particle. The question then is how to construct the new J^2, J_z eigenstates $|jm\rangle$ from $|j_1m_1\rangle$ and $|j_2m_2\rangle$.

The z-components add simply:

$$m=m_1+m_2$$

and j has possible values of

$$j=|j_1-j_2|,|j_1-j_2|+1,\ldots,j_1+j_2-1,j_1+j_2$$

Example 1: Mesons are colorless bound states of quarks and anti-quarks, $q\bar{q}$. If a $q\bar{q}$ pair is in a bound state with zero orbital angular momentum, what are the possible values of the total spin s?

$$s_q = \frac{1}{2}, \qquad s_{\bar{q}} = \frac{1}{2}, \qquad l = 0$$

$$\Rightarrow \qquad s = \begin{cases} \frac{1}{2} - \frac{1}{2} = 0 \\ \frac{1}{2} + \frac{1}{2} = 1 \end{cases}$$

Therefore, $s = 0$ and $s = 1$ are possible.

Example 2: In the quark model, baryons are colorless bound states of three quarks, qqq. If three quarks are combined with zero orbital angular momentum, what are the possible values for the total spin?

To add three or more angular momenta, combine the first two, then add the third, etc…

$$s_1 = \frac{1}{2}, \qquad s_2 = \frac{1}{2}, \qquad s_3 = \frac{1}{2}, \qquad l = 0 \qquad \Rightarrow \qquad s_{12} = 0,1$$

$$\text{if } s_{12} = 0, \text{ then } s = s_3 = \frac{1}{2}$$

$$\text{if } s_{12} = 1, \text{ then } s = \begin{cases} 1 - \frac{1}{2} = \frac{1}{2} \\ 1 + \frac{1}{2} = \frac{3}{2} \end{cases}$$

And that's all there is to know about what you get when you add to angular momenta J_1 and J_2. The only remaining question is what are the amplitudes for getting one of the final values

$$|jm\rangle$$

when we add

$$|j_1 m_1\rangle \text{ and } |j_2 m_2\rangle$$

Since these are both valid bases, there is a unitary transformation that connects the two (in 3-space any two orthogonal bases *i,j,k* and *i',j'k'* are connected by a rotation) and we can write

$$|j_1 j_2 jm\rangle = \sum_{m_1, m_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | j_1 j_2 jm\rangle$$

and

$$|j_1 m_1 j_2 m_2\rangle = \sum_{m_1, m_2} |j_1 j_2 jm\rangle \langle j_1 j_2 jm | j_1 m_1 j_2 m_2\rangle$$

The transformation matrix elements

$$\langle j_1 m_1 j_2 m_2 | j_1 j_2 jm \rangle \equiv \langle j_1 m_1 j_2 m_2 | jm \rangle$$

are called the *Clebsch-Gordon coefficients*. The coefficients are real, so that

$$\langle j_1 m_1 j_2 m_2 | jm \rangle = \langle jm | j_1 m_1 j_2 m_2 \rangle$$

These coefficients are the amplitudes for getting j, m out of the addition of $j_1 m_1$ and $j_2 m_2$. We saw with the addition of two spin-½'s that the z-components of the angular momentum add when adding two spins (see pg. 1), so that the Clebsch-Gordon coefficients vanish unless $m = m_1 + m_2$.

Furthermore,

$$\langle j_1 m_1 j_2 m_2 | jm \rangle \neq 0 \text{ if and only if } |j_1 - j_2| \leq j \leq j_1 + j_2$$

For example,

$$|j_1 m_1\ j_2 m_2\rangle = \sum_{j=|j_1-j_2|}^{(j_1+j_2)} C^{j\, j_1 j_2}{}_{m\, m_1 m_2} |j_1\ j_2\ \ j\, m\rangle, \qquad \text{with } m = m_1 + m_2$$

where $C^{j\,j_1 j_2}_{m\,m_1 m_2}$ are Clebsch-Gordan coefficients. Fortunately, we don't have to work out all the coefficients for a given problem. Tables of Clebsch-Gordon coefficients exist in many books (Table 4.7 in Griffiths) and can be found at the Particle Data Group's web site (http://pdg.lbl.gov).

Example 3: Suppose we have an electron in a hydrogen atom in the state $|\ell m_\ell\rangle = |10\rangle$. What are the possible values of its *total* angular momentum, including spin? In this case, we are adding $\ell = 1$ to $s = 1/2$, so we go to the $1 \otimes \frac{1}{2}$ table in the chart (see next page). We know that $m_\ell = 0$, so there are two rows in the table that we need to look at, $m_s = +1/2$ and $m_s = -1/2$. We find

$$\left|10\frac{1}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle$$

$$\left|10\frac{1}{2}-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2}-\frac{1}{2}\right\rangle$$

Note that the two possible states in each case are $\ell + s$ and $\ell - s$, and in each case we have $m = m_\ell + m_s$ only. What this tells us is that if we add the states with $m_\ell = 0$ and $m_s = 1/2$ and measure *J*, we have a probability of 2/3 of measuring $J = 3/2$ and a probability of 1/3 of measuring $J = 1/2$.

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J J …	
	M M …	
m_1 m_2		
m_1 m_2	Coefficients	
⋮ ⋮		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1 m_2		
J; M	1; +1	
+1/2 +1/2	1	
J; M	1; 0	0; 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2
J; M	1; −1	
−1/2 −1/2	1	

1×1/2

m_1 m_2		
J; M	3/2; +3/2	
+1 +1/2	1	
J; M	3/2; +1/2	1/2; +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3
J; M	3/2; −1/2	1/2; −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3
J; M	3/2; −3/2	
−1 −1/2	1	

2×1

m_1 m_2			
J; M	3; +3		
+2 +1	1		
J; M	3; +2	2; +2	
+2 0	1/3	2/3	
+1 +1	2/3	−1/3	
J; M	3; +1	2; +1	1; +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10
J; M	3; 0	2; 0	1; 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10
J; M	3; −1	2; −1	1; −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5
J; M	3; −2	2; −2	
−1 −1	2/3	1/3	
−2 0	1/3	−2/3	
J; M	3; −3		
−2 −1	1		

1×1

m_1 m_2			
J; M	2; +2		
+1 +1	1		
J; M	2; +1	1; +1	
+1 0	1/2	1/2	
0 +1	1/2	−1/2	
J; M	2; 0	1; 0	0; 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3
J; M	2; −1	1; −1	
0 −1	1/2	1/2	
−1 0	1/2	−1/2	
J; M	2; −2		
−1 −1	1		

2×1/2

m_1 m_2		
J; M	5/2; +5/2	
+2 +1/2	1	
J; M	5/2; +3/2	3/2; +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5
J; M	5/2; +1/2	3/2; +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5
J; M	5/2; −1/2	3/2; −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5
J; M	5/2; −3/2	3/2; −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5
J; M	5/2; −5/2	
−2 −1/2	1	

3/2×1/2

m_1 m_2		
J; M	2; +2	
+3/2 +1/2	1	
J; M	2; +1	1; +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4
J; M	2; 0	1; 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2
J; M	2; −1	1; −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4
J; M	2; −2	
−3/2 −1/2	1	

3/2×1

m_1 m_2			
J; M	5/2; +5/2		
+3/2 +1	1		
J; M	5/2; +3/2	3/2; +3/2	
+3/2 0	2/5	3/5	
+1/2 +1	3/5	−2/5	
J; M	5/2; +1/2	3/2; +1/2	1/2; +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6
J; M	5/2; −1/2	3/2; −1/2	1/2; −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2
J; M	5/2; −3/2	3/2; −3/2	
−1/2 −1	3/5	2/5	
−3/2 0	2/5	−3/5	
J; M	5/2; −5/2		
−3/2 −1	1		

3/2×3/2

m_1 m_2				
J; M	3; +3			
+3/2 +3/2	1			
J; M	3; +2	2; +2		
+3/2 +1/2	1/2	1/2		
+1/2 +3/2	1/2	−1/2		
J; M	3; +1	2; +1	1; +1	
+3/2 −1/2	1/5	1/2	3/10	
+1/2 +1/2	3/5	0	−2/5	
−1/2 +3/2	1/5	−1/2	3/10	
J; M	3; 0	2; 0	1; 0	0; 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4
J; M	3; −1	2; −1	1; −1	
+1/2 −3/2	1/5	1/2	3/10	
−1/2 −1/2	3/5	0	−2/5	
−3/2 +1/2	1/5	−1/2	3/10	
J; M	3; −2	2; −2		
−1/2 −3/2	1/2	1/2		
−3/2 −1/2	1/2	−1/2		
J; M	3; −3			
−3/2 −3/2	1			

2×3/2

m_1 m_2				
J; M	7/2; +7/2			
+2 +3/2	1			
J; M	7/2; +5/2	5/2; +5/2		
+2 +1/2	3/7	4/7		
+1 +3/2	4/7	−3/7		
J; M	7/2; +3/2	5/2; +3/2	3/2; +3/2	
+2 −1/2	1/7	16/35	2/5	
+1 +1/2	4/7	1/35	−2/5	
0 +3/2	2/7	−18/35	1/5	
J; M	7/2; +1/2	5/2; +1/2	3/2; +1/2	1/2; +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10
J; M	7/2; −1/2	5/2; −1/2	3/2; −1/2	1/2; −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5
J; M	7/2; −3/2	5/2; −3/2	3/2; −3/2	
0 −3/2	2/7	18/35	1/5	
−1 −1/2	4/7	−1/35	−2/5	
−2 +1/2	1/7	−16/35	2/5	
J; M	7/2; −5/2	5/2; −5/2		
−1 −3/2	4/7	3/7		
−2 −1/2	3/7	−4/7		
J; M	7/2; −7/2			
−2 −3/2	1			

2×2

m_1 m_2					
J; M	4; +4				
+2 +2	1				
J; M	4; +3	3; +3			
+2 +1	1/2	1/2			
+1 +2	1/2	−1/2			
J; M	4; +2	3; +2	2; +2		
+2 0	3/14	1/2	2/7		
+1 +1	4/7	0	−3/7		
0 +2	3/14	−1/2	2/7		
J; M	4; +1	3; +1	2; +1	1; +1	
+2 −1	1/14	3/10	3/7	1/5	
+1 0	3/7	1/5	−1/14	−3/10	
0 +1	3/7	−1/5	−1/14	3/10	
−1 +2	1/14	−3/10	3/7	−1/5	
J; M	4; 0	3; 0	2; 0	1; 0	0; 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5
J; M	4; −1	3; −1	2; −1	1; −1	
+1 −2	1/14	3/10	3/7	1/5	
0 −1	3/7	1/5	−1/14	−3/10	
−1 0	3/7	−1/5	−1/14	3/10	
−2 +1	1/14	−3/10	3/7	−1/5	
J; M	4; −2	3; −2	2; −2		
0 −2	3/14	1/2	2/7		
−1 −1	4/7	0	−3/7		
−2 0	3/14	−1/2	2/7		
J; M	4; −3	3; −3			
−1 −2	1/2	1/2			
−2 −1	1/2	−1/2			
J; M	4; −4				
−2 −2	1				

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

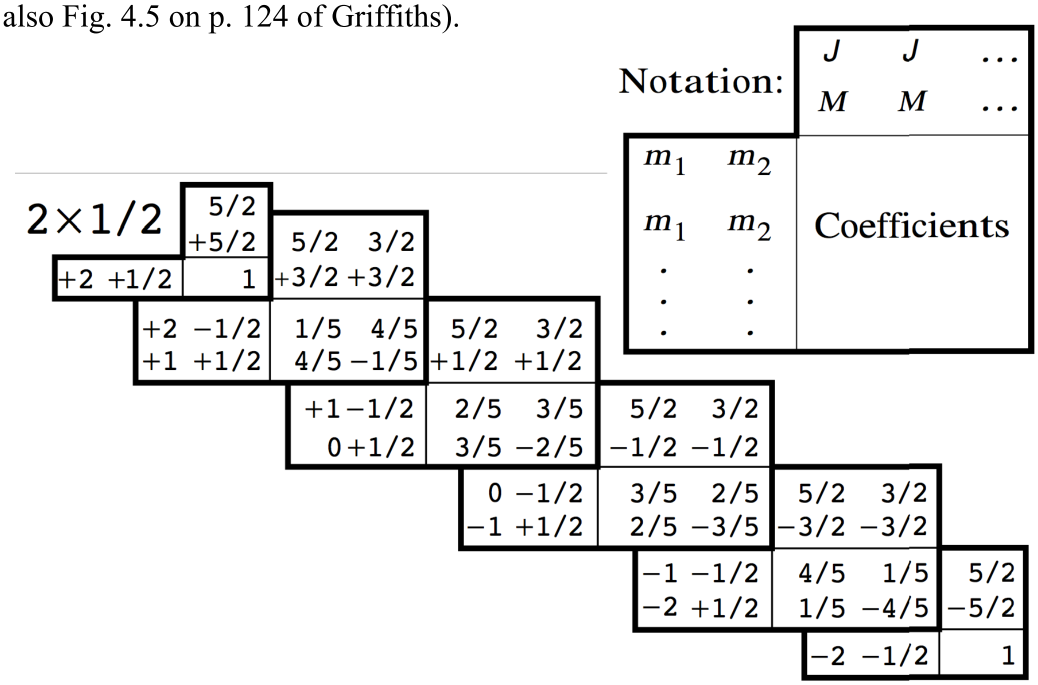
$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

Example 4: An electron in a hydrogen atom is in the $|l, m_l\rangle = |2,1\rangle$ state and in the spin state $|s, m_s\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$. What values of J^2 are possible and what is the probability of measuring each?

$$m = 1 - \frac{1}{2} = \frac{1}{2}, \qquad j = \begin{cases} 2 + \frac{1}{2} = \frac{5}{2} \\ 2 - \frac{1}{2} = \frac{3}{2} \end{cases}$$

To find the decomposition, we look at the $2 \otimes \frac{1}{2}$ coefficients in the Clebsh-Gordan table (see also Fig. 4.5 on p. 124 of Griffiths).



The $2 \otimes \frac{1}{2}$ indicates we are adding $j_1 = 2$ and $j_2 = \frac{1}{2}$, then we use $m = m_1 + m_2 = 1 - \frac{1}{2} = \frac{1}{2}$, and look in the row labeled "+1 $-\frac{1}{2}$". The table lists $\frac{2}{5}$ for $\left|\frac{5}{2}, \frac{1}{2}\right\rangle$ and $\frac{3}{5}$ for $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$. A $\sqrt{\ }$ is implied, so that the table reads:

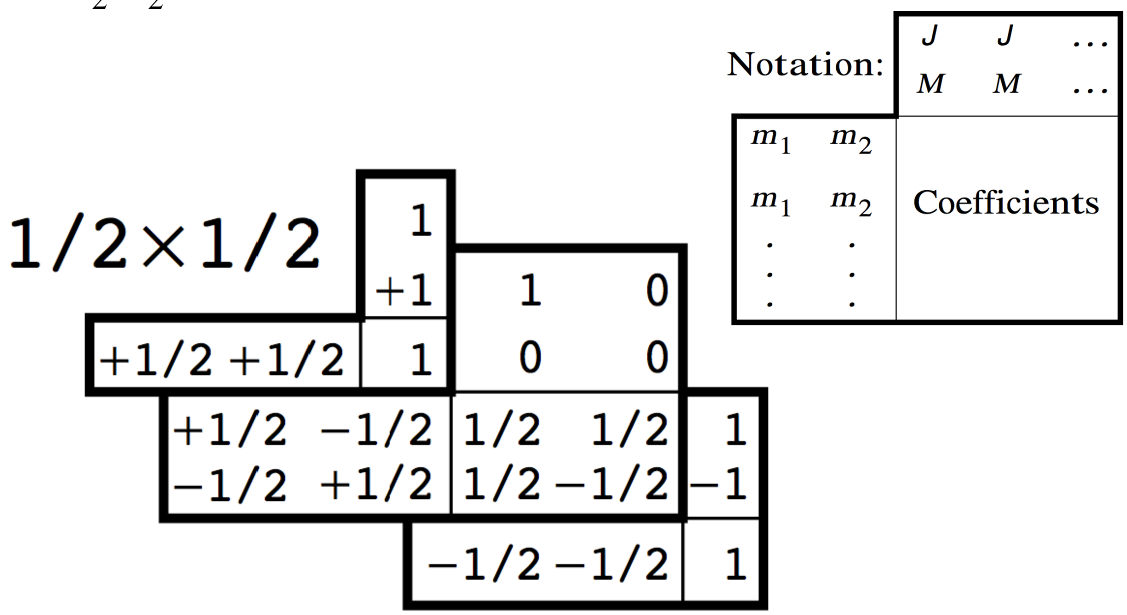
$$|2,1\rangle\left|\frac{1}{2}, \frac{-1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|\frac{5}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

hence

$$P(j = \frac{5}{2}) = \frac{2}{5} \qquad \text{and} \qquad P(j = \frac{3}{2}) = \frac{3}{5}$$

Example 5: Find the Clebsch-Gordan decomposition of the $s=0$ and $s=1$, resulting from combining two spin $s=\frac{1}{2}$ states.

Using the $\frac{1}{2}\otimes\frac{1}{2}$ table:



and reading across the rows tells us

$$|1,1\rangle=\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$|1,0\rangle=\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle+\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$|1,-1\rangle=\left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle$$

$$|0,0\rangle=\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle-\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

We see that there is a triplet of $s=1$ states and a singlet of $s=0$ state. We notice that the $s=1$ states are symmetric under interchange of spins and the $s=0$ state is antisymmetric. More on that later..