Phys 487 Discussion 14 – The Born Approximation

Recap from last discussion: All of our scattering problems rest on the assumption that the scattering potential is central ($V(r)$) and falls off fast enough with r that the wavefunction in the asymptotic $r \to \infty$ region can be treated as that of a free particle. We thus write the total wavefunction of the incoming + outgoing particle as

$$\psi(r) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \quad \text{with} \quad \hbar k = \sqrt{2\mu E}\ .$$

The given quantities are the reduced mass μ of the [beam + target] system and the kinetic energy E of the [beam + target] in the center-of-mass frame. Last week you analyzed the flow of probability represented by the above terms to derived the relation between the scattering amplitude $f(\theta)$ and the differential xsec :

$$\boxed{\frac{d\sigma}{d\Omega} = |\,f(\theta)\,|^2}$$

Today we derived the **Born approximation**. Using Fermi's Golden Rule from 1st-order perturbation theory, we derived this method of calculating the xsec:

$$\boxed{f(\theta) = \frac{\mu}{2\pi\hbar^2} V(q)}$$ where $\boxed{\vec{q} \equiv \vec{k}_i - \vec{k}_f}$ = momentum transfer from beam to target, and

$$\boxed{V(q) = \int d^3r\ V(r) e^{i\vec{q}\cdot\vec{r}}}$$ is the Fourier transform of the potential $V(r)$ created by the target.

This method is applicable when the potential $V(r)$ is << the system's kinetic energy.

Problem 1 : The Born Approximation for two delta functions *Qual Problem* [1]

A free particle of mass m, travelling with momentum p parallel to the z-axis, scatters off the potential

$$V(r) = V_0\left[\delta\left(\vec{r} - \varepsilon\,\hat{z}\right) - \delta\left(\vec{r} + \varepsilon\,\hat{z}\right)\right].$$

(a) Before you do anything else, calculate $q \equiv$ the magnitude of the q-vector in terms of the beam momentum k_i and the scattering angle θ. The definition of the scattering angle is important here : it is the angle between the beam-particle direction $\hat{k}_i$ and the scattered-particle direction $\hat{k}_f$. You will need this relation repeatedly, so put it in a box! Our calculations will be performed in terms of q — notably that Fourier transform we have to do — but at the end, we must express our cross-section in terms of the experimental observables k_i = beam momentum and θ = scattering angle.

▶ GUIDANCE: Hints are in the footnote, but first, the magnitudes of_$\vec{k}_i$ and $\vec{k}_f$ need a little thought. This is **elastic scattering**: there is nowhere for any energy to go, and we are working in the CM frame (via the use of the reduced mass in our equations). If you think through that sentence, you will see that the magnitudes of the incoming and outgoing momenta must be the same : $\boldsymbol{k_i = k_f}$. This matches the classical definition of elastic scattering: initial and final kinetic energies are the same.

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[1] **Q1** **(a)** HINTs from 225: Make! A! Sketch! ☺ of the 3 vectors $\vec{k}_i$, $\vec{k}_f$, and $\vec{q}$ … You must find the magnitude of $\vec{q} \equiv \vec{k}_i - \vec{k}_f$ …
… How do you find the magnitude of a vector? … Dot-it with itself and take the square root … Distribute! The! Dot! ☺ …
… We're doing elastic scattering … so the magnitudes of k_i and k_f are the same … Half-angle formulae are on 1DMath sheet …
Answer: $q^2 = 2k_i^2(1-\cos\theta) \ \to\ q = 2k_i \sin\left(\frac{\theta}{2}\right)$ **(b)** $\frac{d\sigma}{d\Omega} = \frac{m^2 V_0^2}{\pi^2\hbar^4}\sin^2\left(2\varepsilon k \sin^2\frac{\theta}{2}\right)$

▶ FYI: Did you think "typo!" when I called k_i the beam momentum? Good job! ☺ Indeed, $\hbar k$ is momentum … but it is very common to drop "obvious" conversion factors like $\hbar$ and c and refer to wavenumbers as momenta. For high-energy beams of very light particles like electrons, momentum is so close to energy/c that "k" is often just called the "beam energy".

(b) Calculate the differential scattering cross section, $d\sigma / d\Omega$, in the Born approximation.

▶ GUIDANCE: Clearly, you calculate the Fourier transform of the potential, $V(q) = \int d^3r\; V(r) e^{i\vec{q}\cdot\vec{r}}$, then

get $f(\theta) = \frac{\mu}{2\pi\hbar^2} V(q)$, and finally get $\frac{d\sigma}{d\Omega} = |\, f(\theta)\,|^2$. The steps are obvious at least. :-)

One point that requires thought is that your $V(q)$ will end up with "$\vec{q}\cdot\hat{z}$" in a couple of places , and that needs to be expressed in terms of the scattering angle θ. The definition of the scattering angle is important here : it is the angle between the beam-particle direction $\vec{k}_i$ and the scattered-particle direction $\vec{k}_f$. The question tells us that the beam is aligned with the z axis, i.e. $\vec{k}_i = k_i\,\hat{z}$. When you find q_z in terms of q and θ, put it in a box too, as you will need it again.

Problem 2 : The Born Approximation for a Spin-Dependent Potential *Qual Problem* [2]

Consider the non-relativistic scattering of an electron of mass m and momentum k through an angle θ. Calculate the differential cross section in the Born approximation for the spin-dependent potential

$$V(r) = e^{-\mu r^2}\left[\, A + B\,\vec{\sigma}\cdot\vec{r}\,\right]$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the usual vector of Pauli spin matrices and μ, A, B are constants. Assume that the initial-state spin is polarized along the incident direction, and sum over all final-state spins.

▶ GUIDANCE 1: Remember from the first problem: conservation of kinetic energy and those nice relations you had for the q vector's magnitude and z-component.

▶ GUIDANCE 2 = **A CLEVER TRICK**: There is a fabulous way to deal with that $\vec{\sigma}\cdot\vec{r}$ term in the potential. After you have performed your Fourier transform to get $V(q)$, it will become $\vec{\sigma}\cdot\vec{q}$, which you can write as

$$\vec{\sigma}\cdot\vec{q} = \sigma_z q_z + \left(\sigma_x q_x + \sigma_y q_y\right).$$

That doesn't help at all ... but this does:

$$\vec{\sigma}\cdot\vec{q} = \sigma_z q_z + \left(\sigma_+ q_- + \sigma_- q_+\right)$$

Ho! σ_+ and σ_- are the step-up and step-down operators for spin! You know their definitions in terms of σ_x and σ_y … but what are q_- and q_+? They are simple quantities invented to get the ladder operators in there, and I will leave it to you to figure out what they are (algebra! ☺).

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[2] **Q2** $\frac{d\sigma}{d\Omega} = \frac{\pi m^2}{4\hbar^4\mu^3}\, e^{-k^2(1-\cos\theta)/\mu}\left[A^2 + \frac{B^2}{2\mu^2}k^2\left(1-\cos\theta\right)\right]$