Discussion 14 Solution

Problem 2

(a) First of all the probability flux is given as,

$$\vec{j} = \frac{-i\hbar}{2m}\left(\psi^*\vec{\nabla} \quad - \psi\vec{\nabla}\psi^*\right) \tag{1}$$

Let's compute, $\psi^*\vec{\nabla}\psi$ for instance and see what happens (choosing $\vec{k}$ to be in the direction $z = r\cos\theta$)

$$\psi^* = e^{-ikr\cos\theta} + f(\theta)\frac{e^{-ikr}}{r}, \tag{2}$$

$$\vec{\nabla} \quad = \left(\cos\theta e^{ikr\cos\theta} + f(\theta)\left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right)\right)\hat{r} + f'(\theta)\frac{e^{ikr}}{r^2}\hat{\theta}. \tag{3}$$

Therefore, it is easy to see that the cross terms for $\psi^*\vec{\nabla}\psi$ always have,

$$e^{ikr\cos\theta - ikr} = e^{-ikr(1-\cos\theta)}. \tag{4}$$

It can be easily seen that $\psi\vec{\nabla}\psi^*$ will have cross terms that are proportional to,

$$e^{+ikr(1-\cos\theta)}. \tag{5}$$

Hence, following the reasoning stated at the problem we will completely ignore the cross terms in the following problems.

(b) As we are dropping the cross terms the total flux is coming from individual contribution of the wave-function. If we were to notate,

$$\vec{j}[\psi] = \frac{-i\hbar}{2m}\left(\psi^*\vec{\nabla} \quad - \psi\vec{\nabla}\psi^*\right), \tag{6}$$

then the total flux is,

$$\vec{j} = \vec{j}[\psi_{\text{plane}}] + \vec{j}[\psi_{\text{scat}}]. \tag{7}$$

Here we are saying that,

$$\psi_{\text{tot}} = e^{ikr\cos\theta} + f(\theta)\frac{e^{ikr}}{r} \equiv \psi_{\text{plane}} + \psi_{\text{scat}}. \tag{8}$$

It can be easily calculated that,

$$\vec{j}[\psi_{\text{plane}}] = \frac{-i\hbar}{2m}\left((e^{-i\vec{k}\vec{r}})\vec{\nabla}(e^{i\vec{k}\vec{r}}) - (e^{i\vec{k}\vec{r}})\vec{\nabla}(e^{-i\vec{k}\vec{r}})\right) = \frac{\hbar\vec{k}}{m}, \tag{9}$$

and that,

$$\vec{j}[\psi_{\text{scat}}] = \frac{-i\hbar}{2m}\left(\left(f(\theta)\frac{e^{-ikr}}{r}\right)\vec{\nabla}\left(f(\theta)\frac{e^{ikr}}{r}\right) - \left(f(\theta)\frac{e^{ikr}}{r}\right)\vec{\nabla}\left(f(\theta)\frac{e^{-ikr}}{r}\right)\right) = \frac{\hbar k}{m}\frac{|f(\theta)|^2}{r^2}\hat{r}, \tag{10}$$

using,

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}. \tag{11}$$

Hence, the total flux is

$$\frac{\hbar\vec{k}}{m} + \frac{\hbar k}{m}\frac{|f(\theta)|^2}{r^2}\hat{r}. \tag{12}$$

(c) As the problem states so, the second portion is called the scattered flux. In my notation, it is $\vec{j}[\psi_{\text{scat}}]$ Then, it says that,

$$dF = \vec{j}[\psi_{\text{scat}}]\cdot d\vec{A} = \vec{j}[\psi_{\text{scat}}]\cdot\hat{r}r^2 d\Omega. \tag{13}$$

Therefore,

$$dF = \frac{\hbar k}{m}|f(\theta)|^2 d\Omega. \tag{14}$$

Hence,

$$\frac{dF}{d\Omega} = \frac{\hbar k}{m}|f(\theta)|^2. \tag{15}$$

However, as the plane wave flux for one particle is

$$\frac{\hbar k}{m}, \tag{16}$$

So,

$$\frac{\text{particle}}{\text{per small flux}} = \frac{d\sigma}{dF} = \frac{m}{\hbar k}. \tag{17}$$

Therefore,

$$\frac{d\sigma}{d\Omega} = \frac{m}{\hbar k}\frac{\hbar k}{m}|f(\theta)|^2 = |f(\theta)|^2. \tag{18}$$