Phys 487 Discussion 14 – Scattering Introduction

“Scattering problems” are those where this happens:

1. A particle comes in from infinity in some known initial state
   (which is ≈always a plane wave with a given momentum $\hbar\vec{k}$).
2. It passes through a region around the origin where a non-trivial potential $V(\vec{r})$ exists.
3. It goes back off to infinity in some modified state.
   (One other possibility is that the particle is captured or destroyed. This also counts as a scattering problem, but it is an *inelastic* scattering problem. We will mostly stick to elastic scattering.)

Accompanying these steps is one simplifying, yet eminently reasonable, assumption :

★ The scattering potential $V(\vec{r})$ falls off fast enough with r (distance from the origin) that the incoming and outgoing wavefunctions in the **asymptotic** $r \to \infty$ **region** can be treated as those of **free particles**.

It is the **incoming free particle** that distinguishes scattering problems from the bound-state problems that we’ve been doing for many months. Note that you already did scattering problems in 486, they were just in 1D, e.g. “An incoming particle of momentum p is incident on a step potential $V(x) = \ldots$ What happens to it?”
You calculated the reflection and transmission coefficients R and T as ratios of the incoming, reflected, and transmitted probability currents … remember all that? We’re now doing the same sort of thing in **3D**.
The 3D scattering problems we will treat will involve two simplifications: we will treat only

- scattering by **central potentials** $V(r)$, and
- **elastic scattering** where the particle’s kinetic energy is the same in the initial and final states.

Problem 1 : The Asymptotic Wavefunction

Our starting point is to write the **asymptotic form** of the **in**coming + **sc**attered wavefunctions as follows:

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} \psi_{\text{in}} + \psi_{\text{sc}} = e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r} \quad \text{with} \quad \hbar k = \sqrt{2\mu E} \; .$$

The given quantities are the reduced mass μ of the [beam + target] system and the kinetic energy E of the [beam + target] in the center-of-mass frame. By **beam,** we mean the incoming particle; by **target**, we mean whatever is causing the scattering potential $V(r)$ that our beam will encounter in the vicinity of the origin.
As in our 1D scattering problems from 486, we are <u>ignoring the overall normalization</u> of the wavefunction as we will only be interested in the <u>ratio</u> of the scattered wave ψ_{sc} to the incoming wave ψ_{in}. After a few steps of derivation, this ratio will give us the **scattering cross-section** for a given $V(r)$, which is our ultimate goal.

The incoming wave $\psi_{\text{in}} = e^{ikz}$ is very familiar; we instantly recognize it as representing a free particle of momentum $\vec{p} = \hbar k\,\hat{z}$. Our primary assumption ★ above is that $V \approx 0$ if we are far enough away from the origin, so the total energy of the incoming particle is just its kinetic energy at infinity: $E = \hbar^2 k^2 / 2\mu$.
Next, we are assuming <u>elastic scattering</u>, so the energy of the scattered particle must <u>also</u> be $E = \hbar^2 k^2 / 2\mu$.
Your task is to show that the unfamiliar **spherical wave** presented above for the scattered particle,

$$\psi_{\text{sc}} = f(\theta,\phi)\,\frac{e^{ikr}}{r} \, ,$$

does indeed represent a particle of energy $E = \hbar^2 k^2 / 2\mu$ in the region $r \to \infty$. GUIDANCE: Plug ψ_{sc} into the Schrödinger Equation for the case $V = 0$ and apply the asymptotic limit $r \to \infty$ to drop one of the terms. If you’re worried that the term you’re discarding might have an infinity somewhere in all those angle derivatives, identify the part of the Laplacian that is the operator $L^2/\hbar^2$, and realize that you can write any $f(\theta,\phi)$ as a linear combination of the spherical harmonics.

Problem 2 : Currents and Cross-Sections

Now that we have sanity-checked our asymptotic waveform for elastic scattering, let's put it in a box:

$$\boxed{\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}} = \psi_{\text{in}} + \psi_{\text{sc}}$$

Our ultimate goal is to figure out how to calculate a scattering cross-section from this. The cross-section is the 3D version of the R and T coefficients of 1D scattering. Well, R and T were defined as ratios of **probability currents**. Let's follow the same approach to express our asymptotic waveform in terms of real-world measurable things, like the electric current carried by the beam, and the number of scattered events per second measured by a detector placed at a certain angle.

(a) Calculate the probability current

$$\vec{j} = \text{Re}\left[\psi^* \frac{\hat{\vec{p}}}{m}\psi\right] = \frac{-i\hbar}{2\mu}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

for the asymptotic wavefunction in the box above. You will get cross-terms that involve *both* the incoming and scattered waves. These will be hard to interpret if they survive … Show that the cross-terms all contain a term

$$e^{\pm ikr(1-\cos\theta)}.$$

▶ As $r \to \infty$ these terms all average to zero and can be ignored. Hurray!

(b) After dropping the exponential terms in (a), show that, up to terms of order $1/r^2$,

$$\vec{j} = \frac{\hbar\vec{k}}{m} + \frac{\hbar k}{m}\frac{|f(\theta,\phi)|^2}{r^2}\hat{r}.$$

▶ This form is easy to interpret: the first term is the current in the direction of the incoming beam, and the second is the current in the direction of the scattered particle (since $\hat{r}$, by construction, always points from the origin toward the points at angles θ & ϕ). Clearly the two terms are $\vec{j}_{\text{in}}$ and $\vec{j}_{\text{sc}}$. Hurray again!

(c) The number of particles scattered per unit time into a small detector of area $d\vec{A}$ located at angles (θ,ϕ) is proportional to

$$\vec{j}_{\text{sc}}\cdot d\vec{A} = \vec{j}_{\text{sc}}\cdot\left(\hat{r}\,r^2 d\Omega\right)$$

where j_{sc} is the current associated with the scattered wave. The differential cross section dσ/dΩ is this ratio:

$$\frac{d\sigma(\theta,\phi)}{d\Omega} = \frac{\vec{j}_{\text{sc}}\cdot d\vec{A}/d\Omega}{\vec{j}_{\text{in}}\cdot\hat{z}} = \frac{\#\,\text{particles scattered into area dA @ }(\theta,\phi)\text{ per unit [time}\cdot\text{solid angle]}}{\#\text{ incident particles per unit [time]}}$$

Calculate this ratio of currents, and you will have derived the result we are after:

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2}.$$