1. We are interested in determining the rates for the following transitions:

$$|n\, l\, m\rangle \longrightarrow |n'\, l'\, m'\rangle$$

$$|2\,0\,0\rangle \longrightarrow |1\,0\,0\rangle$$

($2s \to 1s$ transition)

$$\begin{cases} |2\,1\,-1\rangle \longrightarrow |1\,0\,0\rangle \\ |2\,1\,0\rangle \longrightarrow |1\,0\,0\rangle \\ |2\,1\,1\rangle \longrightarrow |1\,0\,0\rangle \end{cases}$$

($2p \to 1s$ transitions)

$(n = 1, 2, \ldots)$

$$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = -\frac{13.6\,\text{eV}}{n^2}$$

$$A_{2s \to 1s} = \frac{\omega_{12}^{3} q^{2}}{3\pi \epsilon_0 \hbar c^{3}} \left| \langle 1 0 0 | \vec{r} | 2 0 0 \rangle \right|^{2}$$

$$A^{(1)}_{2p \to 1s} = \frac{\omega_{12}^{3} q^{2}}{3\pi \epsilon_0 \hbar c^{3}} \left| \langle 1 0 0 | \vec{r} | 2 1 -1 \rangle \right|^{2}$$

$$A^{(2)}_{2p \to 1s} = \frac{\omega_{12}^{3} q^{2}}{3\pi \epsilon_0 \hbar c^{3}} \left| \langle 1 0 0 | \vec{r} | 2 1 0 \rangle \right|^{2}$$

$$A^{(3)}_{2p \to 1s} = \frac{\omega_{12}^{3} q^{2}}{3\pi \epsilon_0 \hbar c^{3}} \left| \langle 1 0 0 | \vec{r} | 2 1 1 \rangle \right|^{2}$$

(i) $\langle 1\,0\,0|\vec{r}|2\,0\,0\rangle$

$$= \underline{\langle 100|x|200\rangle}\,\hat{i} + \underline{\langle 100|y|200\rangle}\,\hat{j} + \underline{\langle 100|z|200\rangle}\,\hat{k}$$

$$\langle 100|x|200\rangle = \langle 100|r\sin\theta\cos\phi|200\rangle$$

$$= \underline{\left(\int_0^{2\pi}\int_0^{\pi}|\underset{\sim}{Y_{00}}(\theta,\phi)|^2\sin^2\theta\cos\phi\,d\theta\,d\phi\right)}\underline{\left(\int_0^{\infty}r^3R_{20}(r)R_{10}(r)\,dr\right)}$$

$$= 0 \quad \left(\underline{\int_0^{2\pi}\cos\phi\,d\phi = 0} \text{ and } Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\right)$$

Similarly, $\langle 100|y|200\rangle = 0$.

$$\langle 100|z|200\rangle = \langle 100|r\cos\theta|200\rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} |Y_{00}(\theta,\phi)|^2 \sin\theta \cos\theta \, d\theta \, d\phi \right) \left(\int_0^{\infty} r^3 R_{20}(r) R_{10}(r) \, dr \right)$$

$$= 0 \quad \left(\int_0^{\pi} \sin\theta \cos\theta \, d\theta = \int_{-1}^{1} u \, du = 0 \right)$$

$\therefore \langle 2\,0\,0 | \vec{r} | 1\,0\,0 \rangle = \vec{0}$ which

means $\boxed{A_{2s \to 1s} = 0}$. So the

lifetime $\boxed{\tau_{2s \to 1s} = \frac{1}{A_{2s \to 1s}} = \infty}$ for

the $2s \to 1s$ transition.

Of course using selection rules it's

a no-brainer that $\langle 1\,0\,0|\vec{r}|2\,0\,0\rangle = 0$.

(ii) $\langle 1\,0\,0|\vec{r}|2\,1\,-1\rangle$

$$= \langle 1\,0\,0|x|2\,1\,-1\rangle\hat{i} + \langle 1\,0\,0|y|2\,1\,-1\rangle\hat{j}$$
$$+ \langle 1\,0\,0|z|2\,1\,-1\rangle\hat{k}$$

$$\langle 1\,0\,0|x|2\,1\,-1\rangle = \langle 1\,0\,0|r\sin\theta\cos\phi|2\,1\,-1\rangle$$

$$= \left(\int_0^{2\pi}\int_0^{\pi}\left(Y_{1-1}(\theta,\phi)\right)\left(Y_{00}(\theta,\phi)\right)^{*}\sin^2\theta\cos\phi\, d\theta\, d\phi\right)\left(\int_0^{\infty} r^3 R_{21}(r)R_{10}(r)\,dr\right)$$

$$= \left(\sqrt{\frac{3}{32\pi^2}}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, e^{i\phi}\cos\phi\, d\theta\, d\phi\right)\left(\int_0^{\infty} r^3 R_{21}(r)R_{10}(r)\,dr\right)$$

$$\left(\int_0^{\pi} \sin^3\theta \, d\theta = \int_0^{\pi} \sin\theta \left(1 - \cos^2\theta\right) d\theta \right.$$

$$\left. = \int_{-1}^{1} \left(1 - u^2\right) du = \left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1} = \frac{4}{3} \right)$$

$$\left(\int_0^{2\pi} e^{i\phi} \cos\phi \, d\phi = \frac{1}{2} \int_0^{2\pi} e^{2i\phi} \, d\phi + \frac{1}{2} \int_0^{2\pi} d\phi \right.$$

$$\left. = \pi \right)$$

$$\left(\int_0^{\infty} r^3 R_{21}(r) R_{10}(r) \, dr = \frac{1}{a^3\sqrt{6}} \int_0^{\infty} \left(\frac{r^4}{a}\right) e^{-3r/2a} \, dr \right.$$

$$= \frac{1}{a^4\sqrt{6}}\int_0^\infty r^4 e^{-3r/2a}\,dr = \frac{1}{a^4\sqrt{6}}\left(4!\left(\frac{2a}{3}\right)^5\right)$$

$$= \frac{2^3\cdot 3}{2^{1/2}3^{1/2}}\left(\frac{2}{3}\right)^5 a = \frac{2^{15/2}}{3^{9/2}}\,a \Bigg)$$

$$\Rightarrow \langle 1\,0\,0|x|2\,1\,{-1}\rangle$$

$$= \left(\frac{3^{1/2}}{2^{5/2}\cancel{\pi}}\right)\left(\frac{2^2}{3}\cancel{\pi}\right)\left(\frac{2^{15/2}}{3^{9/2}}\,a\right) = \frac{2^7}{3^5}\,a$$

$$\langle 1\,0\,0|y|2\,1\,{-1}\rangle$$

$$= \langle 1\ 0\ 0 \,|\, r \sin\theta \sin\phi \,|\, 2\ 1\ {-1} \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \left(Y_{1,-1}(\theta,\phi) \right) \left(Y_{00}(\theta,\phi) \right)^{*} \sin^2\theta \sin\phi \, d\theta \, d\phi \right) \left(\int_0^{\infty} r^3 R_{21}(r) R_{10}(r) \, dr \right)$$

$$= \left(\sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \, e^{i\phi} \underline{\sin\phi} \, d\theta \, d\phi \right) \left(\int_0^{\infty} r^3 R_{21}(r) R_{10}(r) \, dr \right)$$

$$= \left(\frac{3^{1/2}}{2^{5/2}\pi} \right) \left(\frac{2^2}{3} \right) \left(-i\pi \right) \left(\frac{2^{15/2}}{3^{9/2}} a \right)$$

$$= \underline{-i} \, \frac{2^7}{3^5} a$$

$$\langle 1\,0\,0 | z | 2\,1\,-1 \rangle$$

$$= \langle 1\,0\,0 | r\cos\theta | 2\,1\,-1 \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \left(Y_{1-1}(\theta,\phi) \right) \left(Y_{00}(\theta,\phi) \right)^* \sin\theta \cos\theta \, d\theta \, d\phi \right) \left(\int_0^{\infty} r^3 R_{21}(r) R_{10}(r) \, dr \right)$$

$$= 0 \quad \left(\int_0^{\pi} \sin\theta \cos\theta \, d\theta = \int_{-1}^{1} u \, du = 0 \right)$$

$$\therefore \langle 1\,0\,0 | \vec{r} | 2\,1\,-1 \rangle$$

$$= \frac{2^7}{3^5} \left(\hat{i} - i\hat{j} \right) a$$

which means

$$A^{(1)}_{2p\to 1s} = \frac{\omega_{12}^3 q^2}{3\pi\epsilon_0 \hbar c^3}\left|\langle 100|\vec{r}|21\,-1\rangle\right|^2$$

$$= \boxed{\frac{\omega_{21}^3 q^2}{3\pi\epsilon_0 \hbar c^3}\left(\frac{2^{15}}{3^{10}}\right)a^2}$$

$$\Rightarrow \boxed{\tau^{(1)}_{2p\to 1s} = \frac{1}{A^{(1)}_{2p\to 1s}} = \frac{3\pi\epsilon_0 \hbar c^3}{\omega_{12}^3 q^2}\,\frac{3^{10}}{2^{15}a^2}}$$

(iii) $\langle 100|\vec{r}|211\rangle$ involves basically the same integrals as (ii).

$$\Rightarrow \langle 100|\vec{r}|211\rangle$$

$$= -\frac{2^7}{3^5}a\left(\hat{i}+i\hat{j}\right)$$

$$\therefore |\langle 211|\vec{r}|100\rangle|^2 =$$

$$|\langle 21-1|\vec{r}|100\rangle|^2$$

$$\Rightarrow \tau^{(3)}_{2p\to 1s} = \tau^{(1)}_{2p\to 1s}$$

$$= \boxed{\frac{3\pi\epsilon_0\hbar c^3}{\omega_{12}^3 q^2}\ \frac{3^{10}}{2^{15}a^2}}$$

(iv) $\langle 1\,0\,0|\vec{r}|2\,1\,0\rangle$

$$= \langle 1\,0\,0|x|2\,1\,0\rangle\hat{i}$$

$$+ \langle 1\,0\,0|y|2\,1\,0\rangle\hat{j}$$

$$+ \langle 1\,0\,0|z|2\,1\,0\rangle\hat{k}$$

$$\langle 1\,0\,0|x|2\,1\,0\rangle$$

$$= \langle 1\,0\,0|r\sin\theta\cos\phi|2\,1\,0\rangle$$

$$= \left(\int_0^{2\pi}\int_0^{\pi}\left(Y_{10}(\theta,\phi)\right)\left(Y_{00}(\theta,\phi)\right)^{*}\sin^2\theta\cos\phi\,d\theta\,d\phi\right)$$

$$\times\left(\int_0^{\infty} r^3 R_{21}(r)\,R_{10}(r)\,dr\right)$$

$$= 0 = \langle 1\,0\,0|y|2\,1\,0\rangle$$

$$\Bigg(Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \quad \text{and}$$

$$\int_0^{2\pi} \cos\phi \, d\phi = \int_0^{2\pi} \sin\phi \, d\phi = 0)$$

$$\langle 100 | z | 210 \rangle$$

$$= \langle 100 | r\cos\theta | 21\,0 \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \left(Y_{10}(\theta,\phi) \right) \left(Y_{00}(\theta,\phi) \right)^* \sin\theta \cos\theta \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{10}(r) \, dr \right)$$

$$= \left(\frac{\sqrt{3}}{2} \int_0^{\pi} \sin\theta \cos^2\theta \, d\theta \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{10}(r) \, dr \right)$$

$$= \left(\frac{1}{3^{1/2}} \right) \left(\frac{2^{15/2}}{3^{9/2}} a \right) = \frac{2^{15/2}}{3^5} a$$

$$\therefore \langle 1\,0\,0 | \vec{r} | 2\,1\,0 \rangle$$

$$= \frac{2^{15/2}}{3^5} a \, \hat{k}$$

$$\Rightarrow A^{(2)}_{2p\to 1s} = \frac{\omega_{12}^3 q^2}{3\pi\epsilon_0 \hbar c^3} \left|\langle 100|\vec{r}|210\rangle\right|^2$$

$$= \frac{\omega_{12}^3 q^2}{3\pi\epsilon_0 \hbar c^3} \frac{2^{15}}{3^{10}} a^2$$

$$\Rightarrow \boxed{\tau^{(2)}_{2p\to 1s} = \frac{1}{A^{(2)}_{2p\to 1s}} = \frac{3\pi\epsilon_0 \hbar c^3}{\omega_{12}^3 q^2} \frac{3^{10}}{2^{15} a^2}}$$

We see that all the transition rates (and therefore lifetimes) <u>are the same for all $2p \to 1s$ transitions</u>

$$\tau^{(1)}_{2p\to 1s} = \tau^{(2)}_{2p\to 1s} = \tau^{(3)}_{2p\to 1s}$$

$$= \frac{3\pi\epsilon_0 \hbar c^3}{\omega_{12}^3 q^2} \frac{3^{10}}{2^{15} a^2} \equiv \underline{\tau}$$

Note that $\omega_{12} = \dfrac{E_1 - E_2}{\hbar}$

$$= \frac{E_1}{\hbar}\left(1 - \frac{1}{4}\right) = \frac{3E_1}{4\hbar} = \frac{3(13.6\,eV)}{4\hbar}$$

$$\therefore \tau = \frac{3\pi\epsilon_0 \hbar c^3}{q^2}\left(\frac{4\hbar}{3E_1}\right)^3 \frac{3^{10}}{2^{15}a^2}$$

$$\approx \boxed{1.6\ \text{ns}}$$ for the $2p \to 1s$ transitions

Whereas the $2s \to 1s$ transition has

$\boxed{\tau = \infty}$ lifetime.

2. (a) Determining the allowed decay routes means applying the selection rules at each step that allow us to identify the nonzero matrix elements.

$$|300\rangle \xrightarrow[\text{(i)}]{} |n\,l\,m\rangle \xrightarrow[\text{(ii)}]{} |100\rangle$$

(i) $\langle n\, l\, m | \vec{r} | 3\,0\,0 \rangle \begin{pmatrix} n=2 \\ l=0,1 \\ m=-1,0,1 \end{pmatrix}$

$$\boxed{\Delta l = \pm 1 \text{ and } \Delta m_l = 0, \pm 1}$$

Thus, $\underline{\langle 2\,1\,{-1}|\vec{r}|3\,0\,0\rangle}$,

$\langle 2\,1\,0|\vec{r}|3\,0\,0\rangle$, and

$\langle 2\,1\,1|\vec{r}|3\,0\,0\rangle$ are all

non-zero, meaning

$$\boxed{|3\,0\,0\rangle \xrightarrow[(i)]{} \left\{\begin{matrix} |2\,1\,1\rangle \\ |2\,1\,0\rangle \\ |2\,1\,{-1}\rangle \end{matrix}\right\}}$$

are all allowed transitions.

(Notice that $\langle 2\,0\,0|\vec{r}|3\,0\,0\rangle = 0$

so that isn't allowed. Also,

$\langle 100|\vec{r}|300\rangle = 0$ so we can't just go directly to the ground state either.

$$|300\rangle \xcancel{\xrightarrow[(i)]{}} |200\rangle$$

(ii) $\langle 100|\vec{r}|n\,l\,m\rangle \neq 0$

iff $\Delta l = \pm 1$ <u>and</u>

$\Delta m_l = 0, \pm 1$

This means $\langle 1\,0\,0|\vec{r}|2\,1\,{-1}\rangle$,

$\langle 1\,0\,0|\vec{r}|2\,1\,1\rangle$, and

$\langle 1\,0\,0|\vec{r}|2\,1\,0\rangle$ are all <u>non-zero</u>.

Therefore,

$$\boxed{\left\{\begin{array}{c} |2\,1\,1\rangle \\ |2\,1\,0\rangle \\ |2\,1\,{-1}\rangle \end{array}\right\} \xrightarrow[(ii)]{} |1\,0\,0\rangle}$$

are all allowed transitions

(Of course, $\langle 1\,0\,0 | \vec{r} | 2\,0\,0 \rangle = 0$

so this transition isn't allowed.)

Thus, all possible decay routes are

shown below.

$$\boxed{|3\,0\,0\rangle \longrightarrow \left\{ \begin{matrix} |2\,1\,1\rangle \\ |2\,1\,0\rangle \\ |2\,1\,-1\rangle \end{matrix} \right\} \rightarrow |1\,0\,0\rangle}$$

(b) $|3\,0\,0\rangle \rightarrow \left\{ \begin{matrix} |2\,1\,1\rangle \\ |2\,1\,0\rangle \\ |2\,1\,-1\rangle \end{matrix} \right\}$

$$\langle 2\,1\,1|\vec{r}\,|3\,0\,0\rangle$$

$$= \langle 2\,1\,1|x|3\,0\,0\rangle\hat{i}$$

$$+ \langle 2\,1\,1|y|3\,0\,0\rangle\hat{j}$$

$$+ \langle 2\,1\,1|z|3\,0\,0\rangle\hat{k}$$

$$\langle 2\,1\,1|x|3\,0\,0\rangle$$

$$= \langle 2\,1\,1|r\sin\theta\cos\phi|3\,0\,0\rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} (Y_{11}(\theta,\phi))^* (Y_{00}(\theta,\phi)) \sin^2\theta \cos\phi \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(-\sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi} \int_0^{\pi} \sin^3\theta e^{-i\phi} \cos\phi \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(-\frac{1}{\pi}\sqrt{\frac{3}{32}}\left(\frac{4}{3}\right)(\pi)\right)\int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr$$

$$= -\frac{1}{\sqrt{6}}\int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr$$

$$\langle 2\ 1\ 1\ |y|\ 3\ 0\ 0\rangle$$

$$= \langle 2\ 1\ 1\ |r \sin\theta \sin\phi|\ 3\ 0\ 0\rangle$$

$$= \left(\int_0^{2\pi}\int_0^{\pi}(Y_{11}(\theta,\phi))^*(Y_{00}(\theta,\phi))\sin^2\theta \sin\phi\, d\theta\, d\phi\right)$$

$$\times \left(\int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr\right)$$

$$= \left(-\sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, e^{-i\phi} \sin\phi \, d\theta \, d\phi\right)$$

$$\times \left(\int_0^\infty r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(-\sqrt{\frac{3}{32\pi^2}} \left(\frac{4}{3}\right)(-i\pi)\right) \int_0^\infty r^3 R_{21}(r) R_{30}(r) \, dr$$

$$= \frac{i}{\sqrt{6}} \int_0^\infty r^3 R_{21}(r) R_{30}(r) \, dr$$

$$\langle 2\,1\,1 | z | 1\,0\,0 \rangle$$

$$= \langle 2 1 1 | r \cos\theta | 1 0 0 \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \left(Y_{11}(\theta,\phi) \right)^* \left(Y_{00}(\theta,\phi) \right) \sin\theta \cos\theta \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(-\sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi} \int_0^{\pi} \sin^2\theta \cos\theta \, e^{-i\phi} \, d\phi \right) \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= 0$$

$$\boxed{\begin{aligned}&\langle 2\,1\,1|\vec{r}|3\,0\,0\rangle \\ &= \left(-\frac{1}{\sqrt{6}}\hat{i} + \frac{i}{\sqrt{6}}\hat{j}\right)\left(\int_0^\infty r^3 R_{21}(r) R_{30}(r)\,dr\right)\end{aligned}}$$

In the same vein,

$$\langle 2\,1\,-1|\vec{r}|3\,0\,0\rangle$$

$$= \langle 2\,1\,-1|x|3\,0\,0\rangle \hat{i}$$

$$+ \langle 2\,1\,-1|y|3\,0\,0\rangle \hat{j}$$

$$+ \langle 2\ 1\ -1 | z | 3\ 0\ 0 \rangle \hat{k}$$

$$\langle 2\ 1\ -1 | x | 3\ 0\ 0 \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \left(Y_{1-1}(\theta,\phi)\right)^* \left(Y_{00}(\theta,\phi)\right) \sin^2\theta \cos\phi \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(\sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \, e^{i\phi} \cos\phi \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^\infty r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(\sqrt{\frac{3}{32\pi^2}} \right) \left(\frac{4}{3} \right) (\pi) \int_0^\infty r^3 R_{21}(r) R_{30}(r) \, dr$$

$$= \frac{1}{\sqrt{6}} \int_0^\infty r^3 R_{21}(r) R_{30}(r) \, dr$$

$$\langle 2\,1\,-1 | y | 3\,0\,0 \rangle$$

$$= \langle 2\,1\,-1 | r \sin\theta \sin\phi | 3\,0\,0 \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \left(Y_{1-1}(\theta,\phi)\right)^{*} \left(Y_{00}(\theta,\phi)\right) \sin^2\theta \ \sin\phi \ d\theta d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(\sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \ e^{i\phi} \sin\phi \ d\theta d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(\sqrt{\frac{3}{32\pi^2}} \right) \left(\frac{4}{3} \right) \left(-i\pi \right) \int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr$$

$$= -\frac{i}{\sqrt{6}} \int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr$$

$$\langle 2\ 1\ -1 | z | 3\ 0\ 0 \rangle = 0$$

$$\boxed{\langle 2\ 1\ -1 | \vec{r} | 3\ 0\ 0 \rangle = \left(\frac{1}{\sqrt{6}} \hat{i} - \frac{i}{\sqrt{6}} \hat{j} \right) \int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr}$$

$$\langle 2\,1\,0|\vec{r}|3\,0\,0\rangle$$

$$= \langle 2\,1\,0|x|3\,0\,0\rangle\hat{i}$$

$$+ \langle 2\,1\,0|y|3\,0\,0\rangle\hat{j}$$

$$+ \langle 2\,1\,0|z|3\,0\,0\rangle\hat{k}$$

$$\langle 2\,1\,0|x|3\,0\,0\rangle$$

$$= \left(\int_0^{2\pi}\int_0^{\pi}\left(Y_{10}(\theta,\phi)\right)^*\left(Y_{00}(\theta,\phi)\right)\sin^2\theta\cos\phi\,d\theta\,d\phi\right)$$

$$\times\left(\int_0^{\infty} r^3 R_{21}(r)\,R_{30}(r)\,dr\right)$$

$$= 0 = \langle 2\,1\,0 | y | 3\,0\,0 \rangle$$

$$\langle 2\,1\,0 | z | 3\,0\,0 \rangle$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} (Y_{10}(\theta, \phi))^* (Y_{00}(\theta, \phi)) \sin\theta \cos\theta \, d\theta \, d\phi \right)$$

$$\times \left(\int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr \right)$$

$$= \left(\frac{\sqrt{3}}{2} \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \right) \int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr$$

$$= \frac{1}{\sqrt{3}} \int_0^{\infty} r^3 R_{21}(r) R_{30}(r) \, dr$$

$$\Rightarrow \langle 2\,1\,0|\vec{r}|3\,0\,0\rangle$$

$$= \left(\frac{1}{\sqrt{3}}\hat{k}\right)\int_0^\infty r^3 R_{21}(r) R_{30}(r)\,dr$$

$$\therefore \left|\langle 2\,1\,1|\vec{r}|3\,0\,0\rangle\right|^2$$

$$= \left|\langle 2\,1\,-1|\vec{r}|3\,0\,0\rangle\right|^2$$

$$= \left|\langle 2\,1\,0|\vec{r}|3\,0\,0\rangle\right|^2$$

$$= \frac{1}{3}\left|\int_0^\infty r^3 R_{21}(r) R_{30}(r)\,dr\right|^2$$

$$A^{(1)}_{3s\to 2p} = A^{(2)}_{3s\to 2p} = A^{(3)}_{3s\to 2p}$$

$$\equiv \boxed{A_{3s\to 2p} = \frac{\omega_{23}^3 q^2}{3\pi\epsilon_0 \hbar c^3}\left(\frac{1}{3}\right)\left|\int_0^\infty r^3 R_{21}(r) R_{30}(r)\,dr\right|^2}$$

All of the three $3s \to 2p$ transitions are <u>equal</u>, so they must go by $\boxed{1/3}$ via each route.

c) Total decay rate $= 3 A_{3s \to 2p}$

$$= \frac{\omega_{23}^3 q^2}{3\pi\epsilon_0 \hbar c^3} \left| \int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr \right|^2$$

$$\omega_{23} = \frac{E_2 - E_3}{\hbar} = \frac{E_1}{\hbar}\left(\frac{1}{4} - \frac{1}{9}\right)$$

$$\Rightarrow \quad \omega_{23} = \frac{5E_1}{36\hbar}$$

$$\int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr$$

$$= \left(\frac{1}{9a^3\sqrt{2}}\right)\int_0^\infty r^3\left(\frac{r}{a}\right)\left(1-\frac{2}{3}\frac{r}{a}+\frac{2}{27}\left(\frac{r}{a}\right)^2\right)e^{-\frac{5r}{6a}}\,dr$$

$$= \frac{1}{9a^4\sqrt{2}}\int_0^\infty r^4 e^{-\frac{5r}{6a}}\,dr - \frac{\sqrt{2}}{27a^5}\int_0^\infty r^5 e^{-\frac{5r}{6a}}\,dr$$

$$+ \frac{\sqrt{2}}{243a^6}\int_0^\infty r^6 e^{-\frac{5r}{6a}}\,dr$$

$$= \frac{1}{9a^4\sqrt{2}}\left(4!\left(\frac{6a}{5}\right)^5\right) - \frac{\sqrt{2}}{27a^5}\left(5!\left(\frac{6a}{5}\right)^6\right)$$

$$+ \frac{\sqrt{2}}{243a^6}\left(6!\left(\frac{6a}{5}\right)^7\right)$$

$$= \frac{a}{9\sqrt{2}} \left(4! \left(\frac{6}{5}\right)^5 - \frac{2}{3} 5! \left(\frac{6}{5}\right)^6 + \frac{2}{27} 6! \left(\frac{6}{5}\right)^7 \right)$$

$$= \frac{a}{9\sqrt{2}} (4!) \left(\frac{6}{5}\right)^5 \left(1 - \frac{2}{3}(120)\left(\frac{6}{5}\right) + \frac{2}{27}(720)\left(\frac{6}{5}\right)^2 \right)$$

$$= \frac{a}{9\sqrt{2}} \left(\frac{4!\,6^5}{5^6} \right) = \underline{\frac{2^7 3^4}{5^6} \sqrt{2}\, a}$$

$$\therefore\ 3A_{3s \to 2p} = \frac{\omega_{23}^3 q^2}{3\pi\epsilon_0 \hbar c^3} \left| \int_0^\infty r^3 R_{21}(r) R_{30}(r)\, dr \right|^2$$

$$= \frac{q^2}{3\pi\epsilon_0 \hbar c^3} \left(\frac{5E_1}{36\hbar} \right)^3 \left(\frac{2^{15} 3^8}{5^{12}} a^2 \right)$$

$$\tau = \frac{1}{3A_{3s\to 2p}} = \frac{3\pi\epsilon_0\hbar c^3}{q^2}\left(\frac{36\hbar}{5E_1}\right)^3\left(\frac{5^{12}}{2^{15}3^8}a^2\right)$$

$$\Rightarrow \boxed{\tau \simeq 0.158\,\mu s}$$