(a) Positronium can be modeled as effectively the same system as an electron in the hydrogen atom, but with a different effective (reduced) mass

$$\mu = \frac{m_e m_p}{(m_e + m_p)} \approx \frac{m^2}{2m} = \frac{1}{2} m$$

since m_e = mass of electron $\approx$ mass of positron $\equiv m$.

Therefore, we may use the same results as for the hydrogen atom, but with the substitution $m \to \mu$ in all formulae.

The ground state energy of positronium, for example, is

$$E_1' = -\left[\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right] = \frac{\mu}{m}\left[-\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]$$

$$= \frac{\mu}{m} E_1 = \frac{1}{2} E_1 = \frac{1}{2}(-13.6\,\text{eV}) = -6.8\,\text{eV}$$

where $E_1 = -13.6\,\text{eV}$ is the usual hydrogen atom ground state energy.

The binding energy of positronium is therefore

$$E_{bind}' = 0 - E_1' = 6.8\,\text{eV}.$$

Just as described in 1(b), the $n=2 \to n=1$ transition wavelength is

$$\lambda' = -\frac{4}{3}\frac{hc}{E_1}\frac{m}{\mu}.$$

Therefore, for positronium,

$$\lambda' = -\frac{4}{3}\frac{hc}{E_1}\cdot 2 = 2.4\times 10^{-7}\,\text{m}.$$

(b) For spontaneous emission, the decay rate from a state $|a\rangle$ to state $|b\rangle$ is given by

$$A = \frac{\omega_0^3 |\vec{p}|^2}{3\pi\epsilon_0\hbar c^3}. \qquad (1)$$

where

$$\omega_0 = \frac{1}{\hbar}(E_b - E_a) \qquad (2)$$

$$\vec{p} = q\langle a|\vec{r}\,|b\rangle. \qquad (3)$$

The lifetime is related to this by

$$\tau = 1/A.$$

To see how τ changes if we change the electron mass m to the reduced mass μ, we need to determine which of the parameters in Eq. (1) depend on mass.

Clearly, Eq. (2) depends on mass since E_a, E_b are the hydrogen/positronium energies, which are proportional to μ. A little less obvious is that Eq. (3) depends on mass. For hydrogen,

$$|\vec{p}| \propto a_0 = \text{Bohr radius}$$

and the Bohr radius itself is mass dependent:

$$a_0 = \frac{\hbar^2}{me^2}.$$

For a reduced mass $\mu \neq m$ system, the effective Bohr radius is:

$$a_0' = \frac{\hbar^2}{\mu e^2} = \frac{m}{\mu}\frac{\hbar^2}{me^2} = \frac{m}{\mu} a_0 .$$

Therefore, we can see that the ratio of the spontaneous emission lifetimes for positronium and hydrogen are

$$\frac{\tau'}{\tau} = \frac{A}{A'} = \frac{\omega_0^3\,|\vec{p}|^2}{\omega_0'^3\,|\vec{p}'|^2} = \left(\frac{E_0}{E_0'}\right)^3\left(\frac{a_0}{a_0'}\right)^2$$

$$= \left(\frac{m}{\mu}\right)^3\left(\frac{\mu}{m}\right)^2 = \frac{m}{\mu} = 2.$$

Since $\tau = 1.6\text{ns}$, $\tau' = 3.2\text{ns}$.