a) We have $i \neq f$, so

$$C_f(t) = \frac{1}{i\hbar}\int_{t_0}^{t} dt'\; H_{fi}(t')\, e^{i\omega_{fi}t'}$$

Here, t_0 = time the perturbation is turned on $= 0$

$$H_{fi}(t') = \langle f \,|\, H(t') \,|\, i\rangle = \langle f|\hat{V}|i\rangle = V_{fi} \quad \text{for} \quad t>0.$$

Note this is time-independent for $t>0$.

$$C_f(t) = \frac{1}{i\hbar}\int_0^t dt'\; V_{fi}\, e^{i\omega_{fi}t'} = \frac{V_{fi}}{i\hbar}\int_0^t e^{i\omega_{fi}t'}dt' = \frac{V_{fi}}{i\hbar}\left[\frac{e^{i\omega_{fi}t'}}{i\omega_{fi}}\right]_0^t = \frac{V_{fi}}{\hbar\omega_{fi}}\left(1-e^{i\omega_{fi}t}\right)$$

Then $P_{i\to f}(t) = |C_f(t)|^2$

$$= \frac{|V_{fi}|^2}{\hbar^2\omega_{fi}^2}\left(1-e^{i\omega_{fi}t}\right)\left(1-e^{-i\omega_{fi}t}\right)$$

$$= \frac{|V_{fi}|^2}{\hbar^2\omega_{fi}^2}\left(2 - 2\cos(\omega_{fi}t)\right)$$

use $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$= \frac{|V_{fi}|^2}{\hbar^2\omega_{fi}^2}\, 2\left(2\sin^2\left(\frac{\omega_{fi}t}{2}\right)\right)$$

$$= \frac{|V_{fi}|^2}{\hbar^2}\;\frac{\sin^2\left(\frac{\omega_{fi}t}{2}\right)}{\left(\frac{\omega_{fi}t}{2}\right)^2}\; t^2$$

b)

We want to show $\delta(ax) = \frac{\delta(x)}{a}$. We can equivalently say $\delta(x) = a\,\delta(ax)$. To show $a\,\delta(ax) = \delta(x)$, we can prove

① $a\,\delta(ax) = 0$ for $x \neq 0$ ② $a\,\delta(ax) = \infty$ for $x=0$ ③ $\int a\,\delta(ax)\,dx = 1$

① & ② are obvious. As for ③, we have

$$\int_{-\infty}^{\infty} a\,\delta(ax)\,dx \underset{u=ax}{=} \int_{-\infty}^{\infty}\delta(u)\,du = 1$$

Thus, $a\,\delta(ax)$ is a δ-fcn, and so $\delta(ax) = \frac{\delta(x)}{a}$

c) We need to prove ①, ②, and ③ for $\lim_{a\to\infty} \frac{1}{\pi} \frac{\sin^2(ax)}{ax^2}$.

① If $x \neq 0$, $\lim_{a\to\infty} \frac{1}{\pi} \frac{\sin^2(ax)}{ax^2} \leq \lim_{a\to\infty} \frac{1}{\pi x^2 a} \longrightarrow 0$

② If $x=0$, for a finite value of a we have

$$\frac{1}{\pi} \frac{\sin^2(ax)}{ax^2}\Bigg|_{x=0} \underset{\uparrow}{=} \frac{1}{\pi} \frac{a^2 \cancel{x}}{a \cancel{x}} = \frac{a}{\pi}$$

use l'hopital

Thus, $\lim_{a\to\infty} \frac{\sin^2(ax)}{\pi a x^2}\Bigg|_{x=0} = \infty$

③ $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(ax)}{ax^2}\,dx \underset{\substack{\uparrow \\ u=ax}}{=} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(u)}{u^2}\,du = 1$$

Thus, $\lim_{a\to\infty} \frac{1}{\pi} \frac{\sin^2(ax)}{ax^2} = \delta(x)$

d) $$R_{i\to f} = \frac{P_{i\to f}}{t} = \frac{|V_{fi}|^2}{\hbar^2} \frac{\sin^2(\omega_{fi} t/2)}{(\omega_{fi} t/2)^2}\, t.$$ ~~Using (c) w/ $a=\frac{t}{2}$, $x=\omega_{fi}$~~

~~t large~~

$$= \frac{|V_{fi}|^2\, 2\pi}{\hbar^2} \frac{\sin^2(\omega_{fi} t/2)}{\pi \frac{t}{2} (\omega_{fi})^2}.$$ Use (c) w/ $a=\frac{t}{2}$, $x=\omega_{fi}$

$$= \frac{|V_{fi}|^2\, 2\pi}{\hbar^2} \delta(\omega_{fi})$$

$$= \frac{2\pi |V_{fi}|^2}{\hbar^2} \delta\left(\frac{E_f - E_i}{\hbar}\right)$$

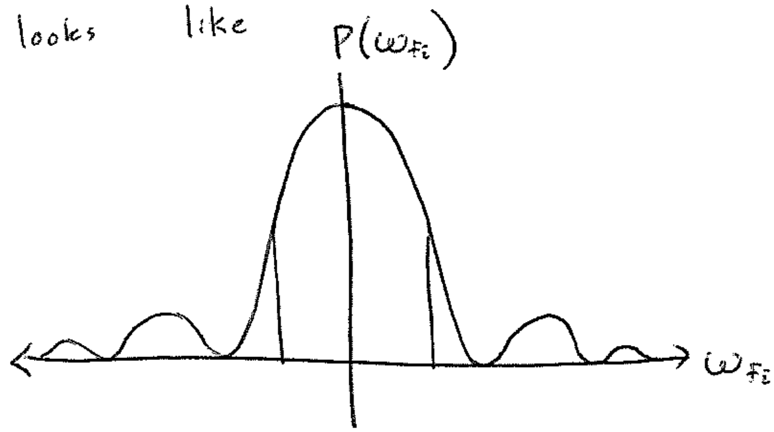
use (b)

$$= \frac{2\pi |V_{fi}|^2}{\hbar} \delta(E_f - E_i)$$

e) We have, for finite t,

$$P_{f\to i}(t) = \frac{|V_{fi}|^2}{\hbar^2} \frac{\sin^2(\omega_{fi}t/2)}{(\omega_{fi}t/2)^2} t^2$$

This looks like



We want to know the approximate width of ω_{fi} → allowed range of E_f.

We have $\frac{\sin^2 x}{x}$ has width π, thus we want

$-\frac{\pi}{2} < x < \pi/2$ for $\frac{\sin^2 x}{x}$ to be "appreciably large". Thus, want $-\frac{\pi}{2} < \frac{\omega_{fi} t}{2} < \frac{\pi}{2}$, or $-\frac{\pi}{t} < \omega_{fi} < \frac{\pi}{t}$

Thus, ω_{fi} gets narrower as $t \to \infty$.

f) t is the length of time $\hat{V}$ was turned on.
In the limit that the perturbation is on for a long time, the only transition w/ nonzero probability is $\omega_{fi} = 0$, thus $E_f - E_i = 0$ and energy is conserved.

g) This is a Heisenberg relation, $\Delta E \Delta t \geq \hbar/2$

h) O.K.