1a) We want to find $\langle 2\ell m | H' | 1\,0\,0 \rangle$.

$$\langle 2\,0\,0 | H' | 1\,0\,0 \rangle = -eE \int \underbrace{\frac{1}{4\sqrt{2\pi}\,a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}}_{\psi_{200}^*} \underbrace{r\cos\theta}_{z} \underbrace{\frac{1}{\sqrt{\pi}\,a_0^{3/2}}e^{-r/a_0}}_{\psi_{100}} r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= -eE \int (r\text{-stuff}) \int (\phi\text{-stuff}) \underbrace{\int_0^{\pi} \cos\theta \sin\theta\, d\theta}_{=0}$$

$$= 0$$

$$\langle 2\,1\,0 | H' | 1\,0\,0 \rangle = -eE \int \frac{1}{4\sqrt{2\pi}\,a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta\; r\cos\theta \frac{1}{\sqrt{\pi}\,a_0^{3/2}} e^{-r/a_0} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{-eE}{4\sqrt{2}\,\pi\, a_0^4} \int_0^{\infty} r^4 e^{-3r/2a_0} dr \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi$$

$$= \frac{-eE}{4\sqrt{2}\,\pi\, a_0^4} \left(24 \left(\frac{2a_0}{3}\right)^5 \right) \left(\frac{2}{3} \right) \left(2\pi \right)$$

$$= -\frac{2^8}{3^5\sqrt{2}}\, eEa_0$$

$$\langle 2\,1\,{\pm 1} | H' | 1\,0\,0 \rangle = \int (r\text{ stuff}) \int (\theta\text{-stuff}) \underbrace{\int_0^{2\pi} e^{\mp i\phi} d\phi}_{=0} = 0$$

b) $$\langle 1\,0\,0 | H' | 1\,0\,0 \rangle = \int (r\text{ stuff}) \underbrace{\int_0^{\pi} \cos\theta \sin\theta\, d\theta}_{=0} \int (\phi\text{ stuff}) d\phi = 0$$

$$\langle 2\,0\,0 | H' | 2\,0\,0 \rangle = \int (r\text{ stuff}) \underbrace{\int_0^{\pi} \cos\theta \sin\theta\, d\theta}_{=0} \int (\phi\text{ stuff}) d\phi = 0$$

$$\langle 2\,1\,0 | H' | 2\,1\,0 \rangle = \int (r\text{ stuff}) \underbrace{\int_0^{\pi} \cos^3\theta \sin\theta}_{=0} \int (\phi\text{ stuff}) = 0$$

$$\langle 2\,1\,{\pm 1} | H' | 2\,1\,{\pm 1} \rangle = \int (r\text{ stuff}) \underbrace{\int_0^{\pi} \cos\theta \sin^3\theta}_{=0} \int (\phi\text{ stuff}) = 0$$

c) If there are only two states, a & b, then the eqn becomes

$$i\hbar \dot{c}_a = H'_{aa} e^{i\omega_{aa}t} c_a(t) + H'_{ab} e^{i\omega_{ab}t} c_b(t)$$

actually zero

$$i\hbar \dot{c}_b = H'_{ba} e^{i\omega_{ba}t} c_a(t) + H'_{bb} e^{i\omega_{bb}t} c_b(t)$$

If additionally, $H'_{aa} = H'_{bb} = 0$, then

$$i\hbar \dot{c}_a = H'_{ab} e^{i\omega_{ab}t} c_b \qquad i\hbar \dot{c}_b = H'_{ba} e^{i\omega_{ba}t} c_a$$

If we define $\omega_{ba} \equiv \omega_0$, then $\omega_{ab} = -\omega_0$, and our eq'ns become

$$\dot{c}_a = \frac{1}{i\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = \frac{1}{i\hbar} H'_{ba} e^{i\omega_0 t} c_a$$

2a) $$\dot{c}_a = \frac{1}{i\hbar}\frac{V_{ab}}{2} e^{-i(-\omega+\omega_0)t} c_b \qquad \dot{c}_b = \frac{1}{i\hbar}\frac{V_{ba}}{2} e^{i(-\omega+\omega_0)t} c_a$$

These are coupled differential eqns, but we can decouple them by taking the derivative of the 2nd equation:

$$\ddot{c}_b = \frac{1}{i\hbar}\frac{V_{ba}}{2} i(-\omega+\omega_0) e^{i(-\omega+\omega_0)t} c_a + \frac{1}{i\hbar}\frac{V_{ba}}{2} e^{i(-\omega+\omega_0)t} \dot{c}_a$$

$$= i(-\omega+\omega_0)\dot{c}_b + \frac{1}{i\hbar}\frac{V_{ab}}{2}\frac{1}{i\hbar}\frac{V_{ba}}{2} c_b \quad , \text{ or}$$

$$\ddot{c}_b + i(\omega-\omega_0)\dot{c}_b + \frac{|V_{ab}|^2}{4\hbar^2} c_b = 0$$

Plugging in the guess $c_b = e^{\lambda t}$ gives

$$\lambda^2 + i(\omega-\omega_0)\lambda + \frac{|V_{ab}|^2}{4\hbar^2} = 0, \quad \text{or}$$

$$\lambda = \frac{-i(\omega-\omega_0) \pm \sqrt{-(\omega-\omega_0)^2 - \frac{|V_{ab}|^2}{\hbar^2}}}{2} = \frac{i}{2}\left(\pm\sqrt{(\omega-\omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}} - (\omega-\omega_0)\right)$$

$$= i\left(\pm\omega_r - \frac{\omega-\omega_0}{2}\right)$$

Thus, our general solution for c_b is

$$c_b = A e^{i\omega_r t - i\frac{(\omega-\omega_0)}{2}t} + B e^{-i\omega_r t - \frac{i(\omega-\omega_0)t}{2}} = e^{i\frac{\omega_0-\omega}{2}t}\left[A e^{i\omega_r t} + B e^{-i\omega_r t}\right]$$

We find A & B from initial conditions. We know $c_b(0)=0$, so we must have $A=-B$, and so $c_b = Ae^{i\frac{\omega_0-\omega}{2}t}\left[e^{i\omega_r t} - e^{-i\omega_r t}\right] = \hat{A}e^{i\frac{(\omega_0-\omega)}{2}t}\sin(\omega_r t)$

We can find c_a from c_b:

$$c_a = \frac{2i\hbar}{V_{ba}} e^{i(\omega-\omega_0)t}\dot{c}_b$$

$$= \frac{2i\hbar}{V_{ba}} e^{i(\omega-\omega_0)t}\hat{A}e^{i\frac{(\omega_0-\omega)}{2}t}\left[\frac{i(\omega_0-\omega)}{2}\sin(\omega_r t) + \omega_r\cos(\omega_r t)\right]$$

We know $c_a(0)=1$, so $\frac{2i\hbar}{V_{ba}}\hat{A}\omega_r = 1$, or $\hat{A} = \frac{V_{ba}}{2i\hbar\omega_r}$.

Thus, our final solution for c_a + c_b is

$$c_a = \frac{2i\hbar}{V_{ba}} e^{i(\omega-\omega_0)t}\left(\frac{V_{ba}}{2i\hbar\omega_r}\right) e^{i\frac{(\omega_0-\omega)}{2}t}\left[\frac{i(\omega_0-\omega)}{2}\sin(\omega_r t) + \omega_r\cos(\omega_r t)\right]$$

$$= e^{i\frac{\omega-\omega_0}{2}t}\left[\cos(\omega_r t) + \frac{i(\omega_0-\omega)}{2\omega_r}\sin(\omega_r t)\right]$$

$$c_b = \frac{V_{ba}}{2i\hbar\omega_r} e^{i\frac{(\omega_0-\omega)}{2}t}\sin(\omega_r t)$$

b) $P_{a\to b}(t) = |c_b|^2 = \frac{|V_{ba}|^2}{4\hbar^2\omega_r^2}\sin^2(\omega_r t)$

$$|c_a|^2 + |c_b|^2 = \left[\cos^2(\omega_r t) + \frac{(\omega_0-\omega)^2}{4\omega_r^2}\sin^2(\omega_r t)\right] + \frac{|V_{ba}|^2}{4\hbar^2\omega_r^2}\sin^2(\omega_r t)$$

just $4\omega_r^2$

$$= \cos^2(\omega_r t) + \left(\frac{\boxed{(\omega_0-\omega)^2 + |V_{ab}|/\hbar^2}}{4\omega_r^2}\right)\sin^2(\omega_r t)$$

$$= \cos^2(\omega_r t) + \sin^2(\omega_r t)$$

$$= 1$$

If $|c_a|^2 + |c_b|^2 \neq 1$, that would mean total probability didn't add to $1 \rightarrow$ not allowed!

c) $P_{a\to b}(t) = \dfrac{|V_{ba}|^2}{4\hbar^2\omega_r^2}\sin^2(\omega_r t)$. We have that

$$\omega_r = \frac{1}{2}\sqrt{(\omega-\omega_0)^2 + \frac{|V_{ab}|^2}{\hbar^2}}$$

$$\cancel{= \frac{|V_{ab}|}{2\hbar}\sqrt{1+\frac{(\omega-\omega_0)^2\hbar^2}{|V_{ab}|^2}}}$$

$$= \frac{1}{2}(\omega-\omega_0)\sqrt{1+\frac{|V_{ab}|^2}{\hbar^2(\omega-\omega_0)^2}}$$

$$\approx \frac{1}{2}(\omega-\omega_0) \quad \text{if} \quad \underline{\frac{|V_{ab}|}{\hbar(\omega-\omega_0)} \ll 1}$$

Then

$$P_{a\to b}(t) = \frac{|V_{ba}|^2}{\hbar^2(\omega-\omega_0)^2}\sin^2\left(\frac{\omega-\omega_0}{2}t\right)$$

~~d) The probabilities return~~

d) The system returns (up to a phase) to $|a\rangle$ when $C_b(t)$ first $=0$ again. Thus, $\omega_r t = \pi$, or $t = \frac{\pi}{\omega_r}$