1) Our Formula for \( C_n(t) \) gives

\[
C_n(t) = \delta_{n0} + \frac{1}{\hbar} \int_{-\infty}^{t} \langle n | H(t') | 0 \rangle e^{\frac{i(E_n - \bar{c} \omega)}{\hbar} t'} dt'
\]

Note that the integration must start before the perturbation is turned on, so we integrate here from \(-\infty\). Simplifying a little gives

\[
C_n(t) = \delta_{n0} - i\frac{\epsilon}{\hbar} \int_{-\infty}^{t} \langle n | \hat{\chi} | 0 \rangle e^{-\frac{\epsilon^2}{2\hbar \omega} t'} dt'
\]

Now, we need to evaluate \( \langle n | \hat{\chi} | 0 \rangle \). We'll use ladder operators.

We know \( \hat{\chi} = \frac{m \omega \hat{x}}{\sqrt{2 \hbar \omega}} \). Thus, \( \hat{\chi}_+ + \hat{\chi}_- = \frac{2m \omega \hat{x}}{\sqrt{2 \hbar \omega}} \), or

\[
\hat{x} = \sqrt{\frac{\hbar}{2m \omega}} (\hat{\chi}_+ + \hat{\chi}_-).
\]

Then

\[
C_n(t) = \delta_{n0} + i\frac{\epsilon}{\hbar} \sqrt{\frac{\hbar}{2m \omega}} \int_{-\infty}^{t} \langle n | (\hat{\chi}_+ + \hat{\chi}_-) | 0 \rangle e^{-\frac{\epsilon^2}{2\hbar \omega} t'} dt'
\]

\[
= \delta_{n0} + i\frac{\epsilon}{\hbar} \frac{1}{\sqrt{2m \omega \hbar}} \int_{-\infty}^{t} e^{-\frac{\epsilon^2}{2\hbar \omega} t'} dt'
\]

\[
= \delta_{n0} + i\frac{\epsilon}{\hbar} \frac{\delta_{n1}}{\sqrt{2m \omega \hbar}} \int_{-\infty}^{t} e^{-\frac{\epsilon^2}{2\hbar \omega} t'} dt'
\]

Now, we're interested in the probability that we end up in the state \( | n \rangle \) at \( t=\infty \). This is simply \( | C_n(\infty) |^2 \). Then

\[
C_n(\infty) = \delta_{n0} + i\frac{\epsilon}{\hbar} \frac{\delta_{n1}}{\sqrt{2m \omega \hbar}} \int_{-\infty}^{\infty} e^{-\frac{\epsilon^2}{2\hbar \omega} t'} dt'
\]

\[
= \delta_{n0} + i\frac{\epsilon}{\hbar} \frac{\delta_{n1}}{\sqrt{2m \omega \hbar}} \frac{\sqrt{\frac{\hbar}{\epsilon^2}}}{\sqrt{2m \omega \hbar \epsilon}} \delta_{n1}
\]

So \( C_0(\infty) = 1 \), \( C_1(\infty) = \frac{i\epsilon}{\hbar} \frac{\delta_{11}}{\sqrt{2m \omega \hbar}} \frac{\sqrt{\frac{\hbar}{\epsilon^2}}}{\sqrt{2m \omega \hbar \epsilon}} \delta_{11} \), \( C_n(\infty) = 0 \), \( n > 1 \).

Now, we want to find probabilities. The probabilities are given by \( |C_n|^2 \). Each of the \( C_n \) are accurate to first order. How accurate are the probabilities?

I'll start with \( C_0 \). If \( \epsilon \) is our small parameter, we know \( C_0 = 1 + A_1 \epsilon + A_2 \epsilon^2 + \ldots \) in general (in our specific case \( A_1 = 0 \) and we don't know \( A_2, A_3, \ldots \), but I'll leave it general).
Then \( |C_0|^2 = (1 + A_1^* \varepsilon + A_1^2 \varepsilon^2 + \ldots) (1 + A_2^* \varepsilon + A_2^2 \varepsilon^2 + \ldots) \)
\[= 1 + (A_1 + A_2^*) \varepsilon + (A_1 A_1^* + A_2 A_2^*) \varepsilon^2 + \ldots \]

We see that to get \( |C_0|^2 \) to first order in \( \varepsilon \), we just need \( A_1 \). To get it to 2nd order in \( \varepsilon \), we need to know \( A_1 A_2 \), etc.

**Punchline:** If we know \( C_0 \) to first order, we get \( |C_0|^2 \) to first order.

What about \( C_n, n > 0 \)? Here we know \( C_n = 0 \) to zeroth order, so
\[C_n = A_1 \varepsilon + A_2 \varepsilon^2 + A_3 \varepsilon^3 + \ldots \]

Now,
\[|C_n|^2 = (A_1^* \varepsilon + A_2^2 \varepsilon^2 + \ldots) (A_1 \varepsilon + A_2 \varepsilon^2 + \ldots) \]
\[= A_1^* A_1 \varepsilon^2 + (A_1 A_1^* + A_2 A_2^*) \varepsilon^3 + \ldots \]

We see that to get \( |C_n|^2 \) to second order, we just need \( A_1 \).

**Punchline:** If we know \( C_n \) to 1st order, we know \( |C_n|^2 \) to 2nd!

So, we can write, for \( n \geq 0 \)
\[P_n = |C_n|^2 = \begin{cases} \frac{q^2 E^2 e^{2\pi\varepsilon}}{2m\omega^2} & n=0 \quad \text{accurate to second order.} \\ 0 & \text{else} \end{cases} \]

We can also write \( P_0 = |C_0|^2 = 1 \), but this is only accurate to 1st order. To get \( P_0 \) to second order, we use the fact that the probabilities must add to 1:
\[P_0 + \frac{q^2 E^2 e^{2\pi\varepsilon}}{2m\omega^2} e^{-\omega^2 \varepsilon^2} = 1 \quad \text{accurate to 2nd order} \]

Or,
\[P_0 = 1 - \frac{q^2 E^2 e^{2\pi\varepsilon}}{2m\omega^2} e^{-\omega^2 \varepsilon^2/2} \]
2) We do the exact same thing. Here,

\[ C_n(t) = \delta_{n0} + \frac{i q E \pi}{\sqrt{2 \pi \hbar}} \int_{-\infty}^{\infty} e^{-i \omega t} \frac{1}{\sqrt{1 + (\omega \delta)^2}} e^{-\omega t} dt' \]

\[ = \delta_{n0} + \frac{i q E \pi}{\sqrt{2 \pi \hbar}} e^{-\omega t} \]

Then \( P_1 = |C_1(t)|^2 = \frac{q^2 E^2 \pi^2}{2m\omega^2} e^{-2\omega t} \leftarrow \text{accurate to 2nd order.} \)

3(b) We want to find the probability of moving into the second energy level. There are four states here, \( |\psi_{100}\rangle, |\psi_{210}\rangle, |\psi_{211}\rangle, \) and \( |\psi_{211-1}\rangle. \)

The probability of being in the second energy level is just the probability of being in the state \( |\psi_{100}\rangle, \) plus the probability of \( |\psi_{210}\rangle, \) plus the probability of \( |\psi_{211}\rangle, \) plus the probability of \( |\psi_{211-1}\rangle. \)

In other words,

\[ P_2 = |C_{100}|^2 + |C_{210}|^2 + |C_{211}|^2 + |C_{211-1}|^2. \]

We need to find each of these \( C_{21m} \)'s.

We have an electric field \( \vec{E} = E e^{-t/\tau} \hat{x}, \) which means a new potential \( V(t) = -qEz e^{-t/\tau}. \)

Thus, our formula for \( C_{21m} \) is

\[ C_{21m}(t) = \frac{i E q}{\hbar} \langle \psi_{21m} | \hat{H} | \psi_{100} \rangle \int_{-\infty}^{\infty} e^{-i \omega t} e^{-\omega t} dt' \]

\[ = \frac{i E q \pi \tau}{\hbar} e^{-\omega t} \langle \psi_{21m} | \hat{\psi} | \psi_{100} \rangle \]

Now we just have to do the brackets. Note \( z = r \cos \theta \) in polar coordinates. Then

\[ \langle \psi_{200} | \hat{\psi} | \psi_{100} \rangle = \int \int \int \frac{1}{4\pi a_0^3} \left( 2 - \frac{r}{a_0} \right) e^{-r/a_0} r \cos \theta \frac{1}{r a_0^2} e^{-\sqrt{a_0} r} r^2 \sin \theta \ dr \ d\phi \ d\theta \]

\[ = 0 \]
\[ \langle \psi_{21} \mid \hat{\mathbf{z}} \mid \psi_{100} \rangle = \int \left( \rho \text{stuff} \right) \int \left( \theta \text{stuff} \right) \int_{0}^{2\pi} e^{i\phi} \, d\phi = 0 \]

\[ \langle \psi_{21} \mid \hat{\mathbf{z}} \mid \psi_{100} \rangle = \int \int \int \frac{1}{\sqrt{2\pi}a_{0}} \frac{1}{\sqrt{2\pi}a_{0}} \frac{1}{\sqrt{2\pi}a_{0}} \left( \frac{1}{\sqrt{2\pi}a_{0}} \right) e^{-r/2a_{0}} \cos \theta \cos \phi \, dr \, d\theta \, d\phi \]

\[ = \frac{1}{4\pi^{3}a_{0}^{3}} \int_{0}^{\infty} r^{4} e^{-3r/2a_{0}} \, dr \int_{0}^{\pi} \cos^{2} \theta \, d\theta \int_{0}^{\pi} \sin \theta \, d\phi \]

\[ = \frac{1}{2\pi a_{0}} \int_{0}^{\infty} r^{4} e^{-3r/2a_{0}} \, dr \int_{0}^{\pi} \cos^{2} \theta \, d\theta \]

\[ = \frac{1}{3\pi a_{0}} \int_{0}^{\infty} r^{4} e^{-3r/2a_{0}} \, dr \]

\[ = \frac{24}{2\pi a_{0}} \left( \frac{2a_{0}}{3} \right)^{5} = \frac{2^{2} a_{0}}{3^{5/2}} \]

Then \( |C_{200}|^{2} = 0 \), \( |C_{211}|^{2} = 0 \), and

\[ |C_{210}|^{2} = \left| \frac{2^{2} a_{0}}{3^{5/2}} e^{-\frac{\omega^{2} T}{2}} \right|^{2} = \frac{e^{2} q^{2} \pi T}{h} e^{-\frac{\omega^{2} T}{2}} \frac{2^{2} a_{0}}{3^{5/2}} \]

Then \( p_{z} = |C_{210}|^{2} \), since all other \( |C_{x\neq0}|^{2} \) are zero.