1) Our formula for $C_n(t)$ gives

$$C_n(t) = \delta_{n0} + \frac{1}{i\hbar}\int_{-\infty}^{t} \langle n|H'(t')|0\rangle e^{i\left(\frac{E_n - E_0}{\hbar}\right)t'}\,dt'$$

Note that the integration must start before the perturbation is turned on, so we integrate here from $-\infty$. Simplifying a little gives

$$C_n(t) = \delta_{n0} - \frac{qE}{i\hbar}\int_{-\infty}^{t} \langle n|\hat{x}|0\rangle e^{-t'^2/\tau^2} e^{i\omega n t'}$$

Now, we need to evaluate $\langle n|\hat{x}|0\rangle$. We'll use ladder operators. We know $\hat{a}_\pm = \frac{m\omega\hat{x} \mp i\hat{p}}{\sqrt{2\hbar m\omega}}$. Thus, $\hat{a}_+ + \hat{a}_- = \frac{2m\omega\hat{x}}{\sqrt{2\hbar m\omega}}$, or

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_+ + \hat{a}_-\right)$. Then

$$C_n(t) = \delta_{n0} + \frac{iqE}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\langle n|(a_+ + a_-)|0\rangle \int_{-\infty}^{t} e^{-t'^2/\tau^2} e^{i\omega n t'}\,dt'$$

$$= \delta_{n0} + \frac{iqE}{\sqrt{2m\omega\hbar}}\langle n|1\rangle \int_{-\infty}^{t} e^{-t'^2/\tau^2} e^{i\omega n t'}\,dt'$$

$$= \delta_{n0} + \frac{iqE}{\sqrt{2m\omega\hbar}}\delta_{n1}\int_{-\infty}^{t} e^{-t'^2/\tau^2} e^{i\omega t'}\,dt'$$

Now, we're interested in the probability that we end up in the state $|n\rangle$ at $t = \infty$. This is simply $|C_n(\infty)|^2$. Then

$$C_n(\infty) = \delta_{n0} + \frac{iqE}{\sqrt{2m\omega\hbar}}\delta_{n1}\int_{-\infty}^{\infty} e^{-t'^2/\tau^2} e^{i\omega t'}\,dt'$$

$$= \delta_{n0} + \frac{iqE\tau\sqrt{\pi}}{\sqrt{2m\omega\hbar}} e^{-\frac{\omega^2\tau^2}{4}}\delta_{n1}$$

So $C_0(\infty) = 1$, $C_1(\infty) = \frac{iqE\tau\sqrt{\pi}}{\sqrt{2m\omega\hbar}} e^{-\frac{\omega^2\tau^2}{4}}$, $C_n(\infty) = 0$, $n > 1$.

Now, we want to find probabilities. The probabilities are given by $|C_n|^2$. Each of the C_n are accurate to first order. How accurate are the probabilities?

I'll start w/ C_0. If ε is our small parameter, we know $C_0 = 1 + A_1\varepsilon + A_2\varepsilon^2 + \dots$ in general (in our specific case $A_1 = 0$ and we don't know $A_2, A_3, \dots$, but I'll leave it general).

Then $|c_0|^2 = (1 + A_1^* \varepsilon + A_2^* \varepsilon^2 + \ldots)(1 + A_1 \varepsilon + A_2 \varepsilon^2 + \ldots)$

$$= 1 + (A_1 + A_1^*)\varepsilon + (A_1 A_1^* + A_2 + A_2^*)\varepsilon^2 + \ldots$$

We see that to get $|c_0|^2$ to first order in ε, we just need A_1. To get it to 2nd order in ε, we need to know A_1 & A_2, etc.

Punchline: If we know c_0 to first order, we get $|c_0|^2$ to first order.

What about c_n, $n > 0$? Here we know $c_n = 0$ to zeroth order, so

$$c_n = A_1 \varepsilon + A_2 \varepsilon^2 + A_3 \varepsilon^3 + \ldots$$

Now,

$$|c_n|^2 = (A_1^* \varepsilon + A_2^* \varepsilon^2 + \ldots)(A_1 \varepsilon + A_2 \varepsilon^2 + \ldots)$$

$$= A_1^* A_1 \varepsilon^2 + (A_1 A_2^* + A_1^* A_2)\varepsilon^3 + \ldots$$

We see that to get $|c_n|^2$ to second order, we just need A_1.

Punchline: If we know c_n to 1st order, we know $|c_n|^2$ to 2nd!

So, we can write, for $n > 0$

$$P_n = |c_n|^2 = \begin{cases} \dfrac{q^2 \mathcal{E}^2 \tau^2 \pi}{2 m \omega \hbar} e^{-\frac{\omega^2 \tau^2}{2}}, & n = 0 \quad \longleftarrow \text{accurate to second order.} \\ 0 & \text{else} \end{cases}$$

We can also write $P_0 = |c_0|^2 = 1$, but this is only accurate to 1st order. To get P_0 to second order, we use the fact that the probabilities must add to 1:

$$P_0 + \frac{q^2 \mathcal{E}^2 \tau^2 \pi}{2 m \omega \hbar} e^{-\frac{\omega^2 \tau^2}{2}} \underset{\uparrow \text{accurate to 2nd order}}{=} 1$$

Or,

$$P_0 = 1 - \frac{q^2 \mathcal{E}^2 \tau^2 \pi}{2 m \omega \hbar} e^{-\omega^2 \tau^2 / 2}$$

2) We do the exact same thing. Here,

$$C_n(\infty) = \delta_{n0} + \frac{iqE}{\sqrt{2m\omega\hbar}}\delta_{n1}\int_{-\infty}^{\infty} \frac{1}{1+(t'/\tau)^2} e^{i\omega t'} dt'$$

$$= \delta_{n0} + \frac{iqE\tau\pi}{\sqrt{2m\omega\hbar}} e^{-\omega\tau}$$

Then $P_1 = |C_1(\infty)|^2 = \dfrac{q^2E^2\tau^2\pi^2}{2m\omega\hbar} e^{-2\omega\tau}$ ← accurate to 2nd order.

3a) We want to find the probability of moving into the second energy level. There are four states here, $|\Psi_{200}\rangle$, $|\Psi_{210}\rangle$, $|\Psi_{211}\rangle$, and $|\Psi_{21-1}\rangle$.

The probability of being in the second energy level is just the probability of being in the state $|\Psi_{200}\rangle$, plus the probability of $|\Psi_{210}\rangle$, plus the probability of $|\Psi_{211}\rangle$, plus the probability of $|\Psi_{21-1}\rangle$.

In other words,

$$P_2 = |C_{200}|^2 + |C_{210}|^2 + |C_{211}|^2 + |C_{21-1}|^2.$$

We need to find each of these $C_{2\ell m}$'s.

We have an electric field $\vec{E} = E e^{-t^2/\tau^2}\hat{z}$, which means a potential energy given by $H'(t) = -qEze^{-t^2/\tau^2}$.

Thus, our formula for $C_{2\ell m}$ is

$$C_{2\ell m}(\infty) = \frac{iEq}{\hbar}\langle\Psi_{2\ell m}|\hat{z}|\Psi_{100}\rangle\int_{-\infty}^{\infty} e^{-t'^2/\tau^2} e^{i\omega t'} dt'$$

$$= \frac{iEq\sqrt{\pi}\tau}{\hbar} e^{-\frac{\omega^2\tau^2}{4}}\langle\Psi_{2\ell m}|\hat{z}|\Psi_{100}\rangle$$

Now we just have to do the brackets. Note $z = r\cos\theta$ in polar coordinates. Then

$$\langle\Psi_{200}|\hat{z}|\Psi_{100}\rangle = \int\!\!\iint \frac{1}{4\sqrt{2\pi}a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}\, r\cos\theta\, \frac{1}{\sqrt{\pi}a_0^{3/2}} e^{-r/a_0} r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int(r \text{ stuff})\, dr \int(\phi \text{ stuff})\, d\phi \underbrace{\int_0^{\pi}\cos\theta\sin\theta\, d\theta}_{=0}$$

$$= 0$$

$$\langle \psi_{21\pm1} | \hat{z} | \psi_{100} \rangle = \int (r \text{ stuff}) \int (\theta \text{ stuff}) \underbrace{\int_0^{2\pi} e^{\mp i\phi} \, d\phi}_{=0} = 0$$

$$\langle \psi_{210} | \hat{z} | \psi_{100} \rangle = \iiint \frac{1}{4\sqrt{2\pi} a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta \; r\cos\theta \frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-r/a_0} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{4\sqrt{2}\pi a_0^4} \int_0^\infty r^4 e^{-3r/2a_0} dr \int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{2\sqrt{2} a_0^4} \int_0^\infty r^4 e^{-3r/2a_0} dr \int_0^\pi \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{1}{3\sqrt{2} a_0^4} \int_0^\infty r^4 e^{-3r/2a_0} dr$$

$$= \frac{24}{3\sqrt{2} a_0^4} \left(\frac{2a_0}{3}\right)^5 = \frac{2^8 a_0}{3^5 \sqrt{2}}$$

Then $|C_{200}|^2 = 0$, $|C_{21\pm1}|^2 = 0$, and

$$|C_{210}|^2 = \left| \frac{i E q \sqrt{\pi} \tau}{\hbar} e^{-\frac{\omega^2 \tau^2}{4}} \frac{2^8 a_0}{3^5 \sqrt{2}} \right|^2 = \frac{E^2 q^2 \pi \tau^2}{\hbar} e^{-\frac{\omega^2 \tau^2}{2}} \frac{2^{15} a_0^2}{3^{10}}$$

Then $P_2 = |C_{210}|^2$, since all other $|C_{2\ell m}|^2$ are zero