Our wf looks like , or $\psi(x) = \begin{cases} \ldots \\ A + \frac{2A}{a}x, & -a/2 < x < 0 \\ 0, & \text{else.} \end{cases}$

Normalization:

We have $1 = \langle\psi|\psi\rangle = \int_{-a/2}^{0} \left(A + \frac{2A}{a}x\right)^2 dx + \int_{0}^{a/2} \left(A - \frac{2A}{a}x\right)^2 dx$

$$= 2\int_0^{a/2} \left(A - \frac{2A}{a}x\right)^2 dx = \frac{A^2 a}{3}, \quad \text{thus} \quad A = \sqrt{\frac{3}{a}}$$

Energy: $\langle\psi|H|\psi\rangle = \langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle + \langle\psi|V(\hat{x})|\psi\rangle$. We'll do these seperately.

$$\langle\psi|V(\hat{x})|\psi\rangle = \int |\psi(x)|^2 (-\alpha\delta(x))dx = -\alpha|\psi(0)|^2 = -\frac{3\alpha}{a}$$

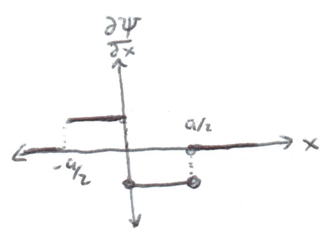
For $\langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle$, we need to be more careful. We know

$\langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle = -\frac{\hbar^2}{2m}\int \psi^*(x)\frac{\partial^2\psi}{\partial x^2}dx$. But what is $\frac{\partial^2\psi}{\partial x^2}$?

See Griffiths 2.24(b) and example 7.3 for more details

~~We have~~ $\cancel{\partial\psi}$

We know $\frac{\partial\psi}{\partial x} = \begin{cases} 0 & x < -a/2 \\ 2A/a & -a/2 < x < 0 \\ -2A/a & 0 < x < a/2 \\ 0 & a/2 < x \end{cases}$



Thus, $\frac{\partial\psi}{\partial x}$ is discontinuous. The derivative of a discontinuous function at the point of discontinuity is a δ-function times the amount the function jumps. Thus, in our case,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{2A}{a}\delta\left(x + \frac{a}{2}\right) - \frac{4A}{a}\delta(x) + \frac{2A}{a}\delta(x - a/2), \quad \text{and}$$

$$\langle\psi|\frac{p^2}{2m}|\psi\rangle = -\frac{\hbar^2}{2m}\int \psi^*(x)\left[\frac{2A}{a}\delta\left(x + \frac{a}{2}\right) - \frac{4A}{a}\delta(x) + \frac{2A}{a}\delta(x - a/2)\right]dx$$

$$\cancel{= -\frac{\hbar^2}{2m}\int \psi^*}$$

$$= -\frac{\hbar^2}{2m}\left[\frac{2A}{a}\psi^*\left(\frac{-a}{2}\right) - \frac{4A}{a}\psi^*(0) + \frac{2A}{a}\psi^*\left(\frac{a}{2}\right)\right] = \frac{4\hbar^2 A^2}{2ma} = \frac{6\hbar^2}{ma^2}$$

Thus $\langle\psi|H|\psi\rangle = \frac{6\hbar^2}{ma^2} - \frac{3\alpha}{a}$

To minimize, $0 = \partial_a \langle\psi|H|\psi\rangle = -\frac{12\hbar^2}{ma^3} + \frac{3\alpha}{a^2}$, or $a = \frac{4\hbar^2}{m\alpha}$. Thus,

$$\langle\psi|H|\psi\rangle_{min} = \frac{6\hbar^2}{m}\left(\frac{m\alpha}{4\hbar^2}\right)^2 - 3\alpha\left(\frac{m\alpha}{4\hbar^2}\right) = \frac{3}{8}\frac{m\alpha^2}{\hbar^2} - \frac{3}{4}\frac{m\alpha^2}{\hbar^2} = -\frac{3}{8}\frac{m\alpha^2}{\hbar^2}$$