1) We know the ground state must $\to 0$ as $x \to \infty$ and $x \to 0$. Thus, I'll guess $\psi(x) = Axe^{-\alpha x}$. Note that there are <u>many</u> possible guesses; anything like $Ax^n e^{-\alpha x^m}$ is a valid guess, for example.

Now, I have to choices: I either normalize ψ and then minimize $\langle\psi|H|\psi\rangle$, or I don't normalize ψ and minimize $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$. They're both approximately the same amount of work. I'll do the first one.

To normalize, I write

$$1 = \langle\psi|\psi\rangle = A^2 \int_0^\infty x^2 e^{-2\alpha x}\,dx = \frac{A^2}{4\alpha^3} \Rightarrow A = 2\alpha^{3/2}$$

Thus, $\psi(x) = 2\alpha^{3/2} x e^{-\alpha x}$

Now, I compute $\langle\psi|H|\psi\rangle$:

$$\langle\psi|H|\psi\rangle = 4\alpha^3 \int_0^\infty \left[-\frac{\hbar^2}{2m} xe^{-\alpha x}\left(\frac{\partial^2}{\partial x^2} xe^{-\alpha x}\right) + Fx^3 e^{-2\alpha x}\right] dx$$

$$= \underbrace{4\alpha^3 \int_0^\infty \frac{\hbar^2}{2m}\left|\frac{\partial}{\partial x} xe^{-\alpha x}\right|^2 dx}_{1} + \underbrace{4\alpha^3 F \int_0^\infty x^3 e^{-2\alpha x}\,dx}_{2}$$

$$1 = \frac{2\alpha^3\hbar^2}{m}\int_0^\infty \left(-\alpha x e^{-\alpha x} + e^{-\alpha x}\right)^2 dx$$

$$= \frac{2\alpha^3\hbar^2}{m}\int_0^\infty \left[\alpha^2 x^2 e^{-2\alpha x} - 2\alpha x e^{-2\alpha x} + e^{-2\alpha x}\right] dx$$

$$= \frac{2\alpha^3\hbar^2}{m}\left[\frac{1}{4\alpha} - \frac{1}{2\alpha} + \frac{1}{2\alpha}\right]$$

$$= \frac{\alpha^2\hbar^2}{2m}$$

$$2 = 4\alpha^3 F\left(\frac{3}{8\alpha^4}\right) = \frac{3F}{2\alpha}$$

Thus,

$$\langle\psi|H|\psi\rangle = \frac{\alpha^2\hbar^2}{2m} + \frac{3F}{2\alpha}$$

Now we need to minimize with respect to α.

$$0 = \frac{\partial}{\partial\alpha}\langle\psi|H|\psi\rangle$$

$$= \frac{\alpha\hbar^2}{m} - \frac{3F}{2\alpha^2}, \text{ or } \alpha^3 = \frac{3Fm}{2\hbar^2}, \text{ or } \alpha = \left(\frac{3Fm}{2\hbar^2}\right)^{1/3}$$

Plugging this in to our formula for $\langle \Psi | H | \Psi \rangle$, we get

$$\langle \Psi | H | \Psi \rangle_{min} = \left(\frac{3Fm}{2\hbar^2}\right)^{2/3} \frac{\hbar^2}{2m} + \frac{3}{2} F \left(\frac{2\hbar^2}{3Fm}\right)^{1/3}$$

$$= \frac{1}{2}\left(\frac{3}{2}\frac{F\hbar}{\sqrt{m}}\right)^{2/3} + \left(\frac{3}{2}\frac{F\hbar}{\sqrt{m}}\right)^{2/3}$$

$$= \frac{3}{2}\left(\frac{3}{2}\frac{F\hbar}{\sqrt{m}}\right)^{2/3}$$

$$= \left(\frac{3}{2}\right)^{5/3}\left(\frac{\hbar^2 F^2}{m}\right)^{1/3}$$

2) To first order, perturbation theory gives us the ground state energy of $(H_0 + H')$ as $E_0^{(0)} + E_0^{(1)}$, where $E_0^{(0)} = \langle \Psi_0 | H_0 | \Psi_0 \rangle$, and $E_0^{(1)} = \langle \Psi_0 | H' | \Psi_0 \rangle$. Thus, our first order estimate for E_0 is

$$E_0^{estimate} = \langle \Psi_0 | H_0 | \Psi_0 \rangle + \langle \Psi_0 | H' | \Psi_0 \rangle = \langle \Psi_0 | (H_0 + H') | \Psi_0 \rangle$$

On the other hand, the <u>actual</u> ground state energy of $(H_0 + H')$ is always less than $\langle \Psi | (H_0 + H') | \Psi \rangle$ for <u>any</u> $|\Psi\rangle$:

$$E_0^{actual} \leq \langle \Psi | (H + H_0) | \Psi \rangle$$

In particular, if we plug in Ψ_0 for Ψ, we get

$$E_0^{actual} \leq \langle \Psi_0 | (H_0 + H') | \Psi_0 \rangle = E_0^{estimate}$$

3) First, normalization:

$$1 = \langle \Psi | \Psi \rangle = A^2 \int x^2 e^{-2bx^2} dx = A^2 \frac{\pi^{1/2}}{2^{7/2} b^{3/2}} \quad \text{or} \quad A = \left(\frac{2^7 b^3}{\pi}\right)^{1/4}$$

Now, we compute $\langle \Psi | H | \Psi \rangle$.

$$\langle \Psi | H | \Psi \rangle = \sqrt{\frac{2^7 b^3}{\pi}} \int_{-\infty}^{\infty} \left[\frac{-\hbar^2}{2m}\left(x e^{-bx^2}\right)\frac{\partial^2}{\partial x^2}\left(x e^{-bx^2}\right) + \frac{1}{2} m\omega^2 x^4 e^{-2bx^2} \right] dx$$

$$= \underbrace{\sqrt{\frac{2^7 b^3}{\pi}} \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} x e^{-bx^2} \right|^2 dx}_{①} + \underbrace{\sqrt{\frac{2^7 b^3}{\pi}} \frac{m\omega^2}{2} \int_{-\infty}^{\infty} x^4 e^{-2bx^2}}_{②}$$

$$① = \sqrt{\frac{2^7 b^3}{\pi}} \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left(e^{-bx^2} - 2bx^2 e^{-bx^2} \right)^2 dx$$

$$= \sqrt{\frac{2^5 b^3 \hbar^4}{\pi m^2}} \int_{-\infty}^{\infty} \left(e^{-2bx^2} - 4bx^2 e^{-2bx^2} + 4b^2 x^4 e^{-2bx^2} \right)$$