We know that the eigenstates of the 1D SHO are given by

$$\psi_n(x) = \left(\frac{1}{\pi x_0^2}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x}{x_0}\right) e^{-x_0^2/2x_0^2}, \quad \text{where } H_n = \text{polynomial of degree } n.$$

In 3D, seperation of variables shows that the eigenstates of a 3D SHO are given by $\psi_{n_x n_y n_z}(x,y,z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$ w/ energy $E_{n_x}+E_{n_y}+E_{n_z}$.

Thus, the ground state of the SHO is given by

$$\psi_{000}(x,y,z) = \psi_0(x)\psi_0(y)\psi_0(z), \quad \text{w/ } \psi_0(x) = \left(\frac{1}{\pi x_0^2}\right)^{1/4} e^{-x^2/2x_0^2}$$

We need to calculate the first order correction due to $\frac{U^2}{\hbar\omega}x^2y^2z^2$ and the first and second order corrections ~~to~~ due to $Uxyz$. This will give us answers to 2nd order in U.

$$E_0^{(1)} = \langle \psi_{000} | Uxyz + \frac{U^2}{\hbar\omega}x^2y^2z^2 | \psi_{000}\rangle = U\underbrace{\langle\psi_{000}|xyz|\psi_{000}\rangle}_{①} + \frac{U^2}{\hbar\omega}\underbrace{\langle\psi_{000}|x^2y^2z^2|\psi_{000}\rangle}_{②}$$

$$① = \int dx \int dy \int dz \; \psi_0^*(x)\psi_0^*(y)\psi_0^*(z)\, xyz\, \psi_0(x)\psi_0(y)\psi_0(z)$$

$$= \int dx\, \psi_0^*(x)\, x\, \psi_0(x) \int dy\, \psi_0^*(y)\, y\, \psi_0(y) \int dz\, \psi_0^*(z)\, z\, \psi_0(z)$$

$$= \left(\int dx\, \psi_0^*(x)\, x\, \psi_0(x)\right)^3$$

$$= \left(\frac{1}{x_0\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-x^2/x_0^2} dx\right)^3$$

$$= 0$$

$$② = \int dx\, \psi_0^*(x)\, x^2\, \psi_0(x) \int dy\, \psi_0^*(y)\, y^2\, \psi_0(y) \int dz\, \psi_0^*(z)\, z^2\, \psi_0(z)$$

$$= \left(\int dx\, \psi_0^*(x)\, x^2\, \psi_0(x)\right)^3$$

$$= \left(\frac{1}{x_0\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/x_0^2} dx\right)^3$$

$$= \left(\frac{x_0^2}{2}\right)^3$$

So $E_0^{(1)} = \frac{U^2}{\hbar\omega}\left(\frac{x_0^2}{2}\right)^3 = \frac{U^2 x_0^6}{8\hbar\omega}$

The second-order piece is given by

$$E^{(2)}_{000} = \sum_{n_x n_y n_z > 0} \frac{|\langle \Psi_{n_x n_y n_z} | Uxyz | \Psi_{000} \rangle|^2}{E_{000} - E_{n_x n_y n_z}}$$

This seems bad. It looks like we have to do an <u>infinite</u> number of brackets. But actually, they are <u>all</u> zero except one.

Let's manipulate the expression for the bracket a little:

$$\langle \Psi_{n_x n_y n_z} | xyz | \Psi_{000} \rangle = \int \Psi^*_{n_x}(x)\, x\, \Psi_0(x)\, dx \int \Psi^*_{n_y}(y)\, y\, \Psi_0(y)\, dy \int \Psi^*_{n_z}(z)\, z\, \Psi_0(z)\, dz$$

$$= \langle \Psi_{n_x} | \hat{x} | \Psi_0 \rangle \langle \Psi_{n_y} | \hat{x} | \Psi_0 \rangle \langle \Psi_{n_z} | \hat{x} | \Psi_0 \rangle$$

~~We know that~~ $\cancel{\Psi_0(x) = (\text{degree } 0 \text{ poly})\, e^{-x^2/2x_0^2}}$, ~~and~~

$\cancel{\Psi_1(x) = (\text{degree } 1 \text{ poly})\, e^{-x^2/2x_0^2}}$.

~~Using linear combinations of~~ $\cancel{\Psi_0}$ ~~and~~ $\cancel{\Psi_1}$, ~~then, we can form any function that looks like~~ $\cancel{(\text{degree } 1 \text{ poly})\, e^{-x^2/2x_0^2}}$.

~~In particular,~~ $\cancel{x\Psi_0(x)}$ ~~can be written as a linear combination of~~ $\cancel{\Psi_0 \,\&\, \Psi_1}$.

We know $\Psi_0(x) = \left(\frac{1}{\pi x_0^2}\right)^{1/4} e^{-x^2/x_0^2}$, and $\Psi_1(x) = \left(\frac{1}{\pi x_0^2}\right)^{1/4} \frac{\sqrt{2}x}{x_0} e^{-x^2/x_0^2}$

Thus, $x\Psi_0(x) = \frac{x_0}{\sqrt{2}} \Psi_1(x)$. We then find

$$\langle \Psi_{n_x n_y n_z} | xyz | \Psi_{000} \rangle = \langle \Psi_{n_x} | \hat{x} | \Psi_0 \rangle \langle \Psi_{n_y} | \hat{x} | \Psi_0 \rangle \langle \Psi_{n_z} | \hat{x} | \Psi_0 \rangle$$

$$= \frac{x_0^3}{2^{3/2}} \langle \Psi_{n_x} | \Psi_1 \rangle \langle \Psi_{n_y} | \Psi_1 \rangle \langle \Psi_{n_z} | \Psi_1 \rangle$$

$$= \begin{cases} x_0^3/2^{3/2} & \text{if } n_x = n_y = n_z = 1 \\ 0 & \text{else} \end{cases}$$

Thus,

$$E^{(2)}_{000} = \frac{|\langle \Psi_{111} | Uxyz | \Psi_{000} \rangle|^2}{E_{000} - E_{111}}.$$ Note $E_{000} = \frac{3}{2}\hbar\omega$, $E_{111} = \frac{9}{2}\hbar\omega$

$$\cancel{\frac{U^2 x_0^6}{-3\hbar\omega} = -\frac{U^2 x_0^6}{3\hbar\omega}} = \frac{U^2 x_0^6}{2^3} \frac{1}{\frac{3}{2}\hbar\omega - \frac{9}{2}\hbar\omega} = -\frac{U^2 x_0^6}{24\hbar\omega}$$

Thus, in total, the change in energy is

$$\delta E = \frac{U^2 x^6}{\hbar\omega}\left(\frac{1}{8} - \frac{1}{24}\right) = \frac{U^2 x^6}{12\hbar\omega}$$