a) We know the spacial wavefunction for Hydrogen is $\Psi_{100} = \frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$ in the ground state.

$$E = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$$

There are really <u>two</u> orthogonal w.f. for Hydrogen w/ this energy: $\Psi_{100}\begin{pmatrix}1\\0\end{pmatrix}$ and $\Psi_{100}\begin{pmatrix}0\\1\end{pmatrix}$.

If we put an electron in <u>each</u> state, the total energy will be the sum of the individual energies,

$$E_{tot} = \frac{me^4}{(4\pi\varepsilon_0)^2\hbar^2}$$

and the wavefunction of the two particles is given by the antisymmetric combination

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\Psi(r_1)\begin{pmatrix}1\\0\end{pmatrix}\Psi(r_2)\begin{pmatrix}0\\1\end{pmatrix} - \underline{\Psi}(r_1)\underline{\begin{pmatrix}0\\1\end{pmatrix}}\,\underline{\Psi}(r_2)\underline{\begin{pmatrix}1\\0\end{pmatrix}}\right]$$

swapped both <u>orbital</u> and <u>spin</u> wfs.

$$= \frac{1}{\sqrt{2}\pi a_0^3} e^{-r_1/a_0} e^{-r_2/a_0}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

b) We want to find $\langle\Psi|V|\Psi\rangle$. First, let's find $V|\Psi\rangle$.

Note that $\vec{S}_1 \cdot \vec{S}_2 = (S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z)$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2 + \begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2 + \begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\right]$$

Thus,

$$V|\Psi\rangle = V_0\,\delta^3(\vec{r}_1-\vec{r}_2)\Psi(r_1)\Psi(r_2)\frac{\hbar^2}{4\sqrt{2}}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right] \times \left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$\begin{aligned}
= \frac{V_0\hbar^2}{4\sqrt{2}}\delta^3(\vec{r}_1-\vec{r}_2)\Psi(\vec{r}_1)\Psi(\vec{r}_2)\Bigg[&\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_2\\
&+ \begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_2\\
&+ \begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\begin{pmatrix}1\\0\end{pmatrix}_2\Bigg]
\end{aligned}$$

$$= \frac{V_0\hbar^2}{4\sqrt{2}}\delta^3(\vec{r}_1-\vec{r}_2)\Psi(\vec{r}_1)\Psi(\vec{r}_2)\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\i\end{pmatrix}_1\begin{pmatrix}-i\\0\end{pmatrix}_2 - \begin{pmatrix}-i\\0\end{pmatrix}_1\begin{pmatrix}0\\i\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\-1\end{pmatrix}_2 - \begin{pmatrix}0\\-1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

$$V|\Psi\rangle = \frac{V_0\hbar^2}{4\sqrt{2}}\delta^3(\vec{r}_1-\vec{r}_2)\Psi(\vec{r}_1)\Psi(\vec{r}_2)\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}0\\i\end{pmatrix}_1\begin{pmatrix}-i\\0\end{pmatrix}_2-\begin{pmatrix}-i\\0\end{pmatrix}_1\begin{pmatrix}0\\i\end{pmatrix}_2+\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\-1\end{pmatrix}_2-\begin{pmatrix}0\\-1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

factor out scalars (i)(–i)=1 factor out scalar (–1)

$$= \frac{V_0\hbar^2}{4\sqrt{2}}\delta^3(\vec{r}_1-\vec{r}_2)\Psi(r_1)\Psi(r_2)\left[3\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-3\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

$$= -\frac{3\hbar^2}{4\sqrt{2}}V_0\delta(r_1-r_2)\Psi(r_1)\Psi(r_2)\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

Then we can write

$$\langle\Psi|V\Psi\rangle = \int d^3\vec{r}_1\int d^3\vec{r}_2\ \Psi^*(r_1)\Psi^*(r_2)\left(\frac{-3\hbar^2}{8}\right)V_0\delta(r_1-r_2)\Psi(r_1)\Psi(r_2)$$
$$\times\left[(1\,0)(0\,1)-(0\,1)(1\,0)\right]\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$= \int d^3r\,|\Psi(r)|^4\left(-\frac{3\hbar^2}{8}\right)V_0\,[2]$$

$$= -\frac{3\hbar^2}{4}V_0\int dr\int \sin\theta\,d\theta\int d\phi\ \frac{r^2}{\pi^2a_0^6}e^{-4r/a_0}$$

$$= \frac{-3\hbar^2 V_0}{\pi a_0^6}\int_0^\infty r^2e^{-r/a_0}\,dr$$

$$= -\frac{3\hbar^2V_0}{\pi a_0^6}\left(\frac{a_0^3}{32}\right)$$

$$= -\frac{3\hbar^2V_0}{32\pi a_0^3}$$