a)

Infinite square well problem of "single particle":

$$\begin{cases} \psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \\ E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \end{cases} \quad (n = 1, 2, 3, \ldots)$$

For 2 identical bosons, suppose we know 1 boson is in E_i state and 1 boson is in E_j state (suppose non-degeneracy)

If $E_i \neq E_j$,

$$\psi_{i,j}(x_1, x_2) = \frac{\psi_i(x_1)\psi_j(x_2) + \psi_i(x_2)\psi_j(x_1)}{\sqrt{2}}$$

And the total energy for this state is:

$$E_{i,j} = E_i + E_j$$

If $E_i = E_j$,

$$\psi_{i,i}(x_1, x_2) = \psi_i(x_1)\psi_i(x_2)$$

total energy

$$E_{i,i} = 2E_i$$

① Ground state

The ground state is for "$i=j=1$"

Ground state wave function:

$$\psi_{1,1}(x_1,x_2) = \varphi_1(x_1)\varphi_1(x_2)$$

$$= \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x_1}{a}\right)\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x_2}{a}\right)$$

$$= \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$$

Ground state energy:

$$E_{1,1} = E_1 + E_1 = 2\times\frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{ma^2}$$

② first excited state

The first excited state is for "$i=1, j=2$"

(Because the 2 bosons are identical you can not distinguish "$i=1, j=2$" from "$i=2, j=1$".
You can easily check $\psi_{i,j}(x_1, x_2) = \psi_{j,i}(x_1, x_2)$)

Wavefunction :

$$\psi_{1,2}(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{a}} \sin\frac{\pi x_1}{a} \cdot \sqrt{\frac{2}{a}} \sin\frac{2\pi x_2}{a} + \sqrt{\frac{2}{a}} \sin\frac{\pi x_2}{a} \cdot \sqrt{\frac{2}{a}} \sin\frac{2\pi x_1}{a}\right)$$

$$= \frac{\sqrt{2}}{a}\left(\sin\frac{\pi x_1}{a} \sin\frac{2\pi x_2}{a} + \sin\frac{\pi x_2}{a} \sin\frac{2\pi x_1}{a}\right)$$

energy:

$$E_{1,2} = E_1 + E_2$$

$$= \frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2} = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}$$

b)

① Ground state

First, ground state is non-degenerate!

First order correction:

$$E^{(1)} = \langle \psi_{1,1} | V_{INT} | \psi_{1,1} \rangle$$

$$= \int_0^a \int_0^a \frac{2\sqrt{2}}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \left[-aV_0 \delta(x_1 - x_2)\right] \frac{2\sqrt{2}}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) dx_1 dx_2$$

$$= \int_0^a \left(-\frac{8V_0}{a}\right) \sin\frac{\pi x_2}{a} \sin\frac{\pi x_2}{a} \sin\frac{\pi x_2}{a} \sin\frac{\pi x_2}{a} dx_2$$

$$= \int_0^a \left(-\frac{8V_0}{a}\right) \left(\sin\frac{\pi x_2}{a}\right)^4 dx_2$$

$$\boxed{\text{Set } u \equiv \frac{\pi x_2}{a}}.$$

$$= \int_0^{\pi} \left(-\frac{8V_0}{a}\right) \sin^4 u \left(\frac{a}{\pi}\right) du$$

$$= \int_0^{\pi} \left(-\frac{8V_0}{\pi}\right) \sin^4 u \, du.$$

$$= \left(-\frac{8V_0}{\pi}\right) \times 2 \times \int_0^{\frac{\pi}{2}} \sin^4 u \, du$$

Formula:

Suppose m, n are both odd. stops before you hit "0"

$$\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x \, dx$$

$$= \frac{1 \cdot (n-1)(n-3)(n-5)\cdots \quad (m-1)(m-3)(m-5)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots}$$

i.e. $\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx = \frac{1}{2}$, $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx = \frac{1}{3}$

Suppose m and n are "both even"

throw in a $\left(\times \frac{\pi}{2}\right)$

i.e. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx = \frac{1 \times 1}{4 \times 2} \times \frac{\pi}{2} = \frac{\pi}{16}$

$$E^{(1)} = \left(-\frac{8V_0}{\pi}\right) \times 2 \times \int_0^{\frac{\pi}{2}} \sin^4 u \, du$$

$$= \frac{-16V_0}{\pi} \cdot \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = -\frac{3V_0}{2}$$

② First excited state

$$E^{(1)} = \langle \psi_{1,2} | V_{INT} | \psi_{1,2} \rangle$$

$$= \int_0^a \int_0^a \psi_{1,2}(x_1, x_2)^* \left(-aV_0\,\delta(x_1 - x_2)\right) \psi_{1,2}(x_1, x_2)\, dx_1 dx_2$$

$$= \int_0^a \int_0^a \psi_{1,2}(x_1, x_2) \left(-aV_0\,\delta(x_1 - x_2)\right) \psi_{1,2}(x_1, x_2)\, dx_1 dx_2$$

$$= \int_0^a (-aV_0) \cdot \psi_{1,2}(x_1, x_1)\, \psi_{1,2}(x_1, x_1)\, dx_1$$

$$\boxed{\begin{aligned}\psi_{1,2}(x_1, x_1) &= \frac{\sqrt{2}}{a}\left(\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_1}{a} + \sin\frac{\pi x_1}{a}\sin\frac{2\pi x_1}{a}\right)\\ &= \frac{2\sqrt{2}}{a}\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_1}{a}\end{aligned}}$$

$$= \int_0^a (-aV_0)\, \frac{8}{a^2} \sin^2\frac{\pi x_1}{a} \sin^2\frac{2\pi x_1}{a}\, dx_1$$

$$\boxed{u \equiv \frac{\pi x_1}{a}}$$

$$= \int_0^\pi \left(-\frac{8V_0}{a}\right) \sin^2 u \sin^2 2u \left(\frac{a}{\pi} du\right)$$

$$= \int_0^\pi \left(-\frac{8V_0}{\pi}\right) \sin^2 u \; 4\sin^2 u \cos^2 u\, du$$

$$= \frac{-32V_0}{\pi} \int_0^\pi \sin^4 u \cos^2 u\, du = -\frac{64V_0}{\pi} \int_0^{\frac{\pi}{2}} \sin^4 u \cos^2 u\, du$$

$$= -\frac{64V_0}{\pi} \quad \frac{3\times 1\times 1}{6\times 4\times 2} \times \frac{\pi}{2}$$

$$= -2V_0$$